Honours Analysis 3 Notes (Math 354)

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Contents

§1 Tuesday, September 3, 2013

- Course website: `www.math.mcgill.ca/~jaksic/MATH354-2013.html`
- Assignments once per week; late work not accepted (because solutions are posted).
- Optional tutorials for advanced topics
- Working seminar in mathematical physics: every Wednesday
- Talk to Jaksic in February if you want a job in analysis this summer
- Will start following book closely after a week or two
- Will be very difficult

§1.1 Ch. 9: Metric Spaces General Properties

Definition 1.1 A metric space is a pair (X, d) where X is a nonempty set and d is a function (called the metric),

$$d : X \times X \to [0, \infty)$$

such that the following holds for all $x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

A consequence of (4) and elementary induction is that for $x_1, \ldots, x_n \in X$,

$$d(x_1, x_n) \leq d(x_1, x_2) + \ldots + d(x_{n-1}, x_n)$$

(Exercise)

Examples

1. Analysis I,II: $X = \mathbb{R}$, $d(x, y) = |x - y|$
2. $X = \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. $d(x, y) = \left(\sum_{k=1}^{n}(x_k - y_k)^2\right)^{1/2}$
3. $X = \mathbb{R}^n$, $d(x, y) = \max_{1\leq k\leq n} |x_k - y_k|$
4. $C([a, b])$ is the vector space of all continuous functions

   $$f : [a, b] \to \mathbb{R}$$

   where

   $$(f + g)(x) = f(x) + g(x)$$
   $$(\alpha f)(x) = \alpha f(x)$$

   Then $\max_{x\in[a,b]} |f(x)|$ is a finite number, and $\exists\, x_0 \in [a, b]$ such that $\max_{x\in[a,b]} |f(x)| = |f(x_0)|$
   Let

   $$d(f, g) = \max_{x\in[a,b]} |f(x) - g(x)|$$

5. $C([a, b])$, $d(f, g) = \left(\int_a^b |f(x) - g(x)|^2 \,\mathrm{d}x\right)^{1/2}$
6. (Discrete metric). Let X be a non-empty set. Let

   $$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

   Then d is a metric on X: Is $d(x, y) \leq d(x, z) + d(z, y)$? If $x = y$, obvious. If $x \neq y$, we have three cases:

   1. $z = x$, then $z + y$; $1 \leq 1$
   2. $z = y$, then $z + x$, $1 \leq 1$
   3. $z \neq x, z \neq y$, $1 \leq 2$

Examples 1 through 5 come from a metric that is induced by a norm. And in examples 2 and 5, the norm is induced by an inner product.

Definition 1.2 Let X be a vector space over $\mathbb{R}$. A function

$$X \to x \mapsto \|x\| \in [0, \infty)$$

is called a norm on X if the following holds for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0_X$
3. $\|\alpha x\| = |\alpha|\|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

For vector spaces over complex numbers, the only thing that changes is $|\alpha|$ is the norm of the complex number α.

Metric induced by a norm: Given normed space $(X, \|\cdot\|)$, the function

$$d(x,y) = \|x-y\|$$

is a metric on X. d is called a metric induced by a norm.

$$d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1|\|y-x\| = d(y,x)$$

and

$$d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y)$$

Example 1.3 $X = \mathbb{R}$, $|x|$ is a norm on $\mathbb{R}$

Example 1.4 $X = \mathbb{R}^n$, $\|x\| = \left(\sum_{k=1}^n x_k^2\right)^{1/2}$, then $d(x,y) = \left(\sum_{k=1}^n (x_k - y_k)^2\right)^{1/2}$.

Example 1.5 $\mathbb{R}^n$, $\|x\| = \max_{1\leq k\leq n} |x_k|$ This is a norm on $\mathbb{R}^n$. Then $d(x,y) = \|x-y\| = \max_{1\leq k\leq n} |x_k - y_k|$ is a metric.

Triangle inequality: $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. $x+y = (x_1+y_1, \ldots, x_n+y_n)$.

$$\|x\| = \max_{1\leq k\leq n} |x_k| = |x_{k'}|$$

for some $k' \in \{1, \ldots, n\}$ (we know this).

Similarly,

$$\|y\| = \max_{1\leq k\leq n} |y_k| = |y_{k''}|$$

for some $k'' \in \{1, \ldots, n\}$.

So

$$\|x+y\| = \max_{1\leq k\leq n} |x_k + y_k| = |x_{k'''} + y_{k'''}| \leq |x_{k'''}| + |y_{k'''}| \leq |x_{k'}| + |y_{k''}| = \|x\| + \|y\|$$

Example 1.6 $C([a,b])$. $\|f\| = \max_{x\in[a,b]} |f(x)|$.

$$d(f,g) = \|f-g\| = \max_{x\in[a,b]} |f(x) - g(x)|$$

1. $\|f\| \geq 0$ obvious
2. $\|f\| = 0 \iff \max_{x\in[a,b]} |f(x)| = 0$. So $0 = \max_{x\in[a,b]} |f(x)| \geq |f(x_1)| \geq 0$. True for all x_1 so $f(x) = 0$ for $x \in [a,b]$.
3. $\|\alpha f\| = \max_{x\in[a,b]} |\alpha f(x)| = \max_{x\in[a,b]} |\alpha||f(x)| = |\alpha||f(x_0)| = |\alpha| \max_{x\in[a,b]} |f(x)| = |\alpha|\|f\|$
4. Triangle: $f, g \in C([a,b])$,

$$\|f\| = \max_{x\in[a,b]} |f(x)| = |f(x_0)|$$

$$\|g\| = \max_{y\in[a,b]} |g(y)| = |g(y_0)|$$

$$\|f+g\| = \max_{x\in[a,b]} |f(x)+g(x)| = |f(x_0)| + |g(y_0)| \leq |f(x_0)| + |g(y_0)| = \|x\| + \|y\|$$

§2 Thursday, September 5, 2013

§2.1 Review and introduction to metric spaces

If $\|\cdot\|$ is a norm on a vector space, then $d(x,y) = \|x-y\|$ is a metric induced by the norm. Examples 1-5 from last day are metrics induced by norms.

Example 5 $C([a,b])$, $\|f\| = \left(\int_a^b |f(x)|^2 \, \mathrm{d}x\right)^{1/2}$. This is a norm; we will check after introducing inner product spaces.

Example 6 If X is a nontrivial vector space and d is the discrete metric on X,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

then d is *not* induced by a norm.

To see this, assume $d(x, y) = \|x - y\|$.

If $x \neq y$, $d(x, y) = 1$. For any scalar $\alpha \neq 0$, $\alpha x \neq \alpha y$. But

$$d(\alpha x, \alpha y) = 1 = \|\alpha x - \alpha y\| = |\alpha|\|x - y\| = |\alpha| d(x, y) = |\alpha|$$

Thus, $|\alpha| = 1$ for all α; contradiction.

§2.2 Induced norms

There is a special class of norms that have Euclidean geometry. These are norms induced by an inner product.

Definition 2.1 Let X be a vector space over $\mathbb{R}$. A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called an inner product on X if the following holds for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{R}$:

1. $\langle x, x \rangle \geq 0$
2. $\langle x, x \rangle = 0 \iff x = 0_X$
3. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
4. $\langle x, y \rangle = \langle y, x \rangle$

Remark.

$$\langle \alpha y + \beta z, x \rangle = \langle x, \alpha y + \beta z \rangle = \alpha \langle y, x \rangle + \beta \langle x, z \rangle = \alpha \langle y, x \rangle + \beta \langle z, x \rangle$$

Remark. If X is a vector space over $\mathbb{C}$ and $\langle \cdot, \cdot \rangle$ is a complex inner product, then (4) changes.

$$(4): \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

so then

$$\langle \alpha y + \beta z, x \rangle = \overline{\langle x, \alpha y + \beta z \rangle} = \overline{\alpha \langle x, y \rangle + \beta \langle x, z \rangle} = \overline{\alpha}\overline{\langle x, y \rangle} + \overline{\beta}\overline{\langle x, z \rangle} = \overline{\alpha} \langle y, x \rangle + \overline{\beta} \langle z, x \rangle$$

i.e. the complex inner product is anti-linear in the first argument.

Conventions: In pure math, the complex inner product is linear in the first argument and anti-linear in the second; in applied math and physics, the reverse is true.

Proposition 2.2 Cauchy Schwarz: $\forall x, y \in X$, $|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle}\sqrt{\langle y, y \rangle}$

Proof. If $x = 0_X$ or $y = 0_X$, the statement is obvious as both sides are zero.

$$\langle x, 0_X \rangle = \langle x, 0 \cdot 0_X \rangle = 0 \langle x, 0_X \rangle = 0$$

So we can assume $x \neq 0_X$ and $y \neq 0_X$.

Define

$$f(t) := \langle x + ty, x + ty \rangle$$

for $t \in \mathbb{R}$. Then $f(t) \geq 0$ for all t.

By linearity,

$$\begin{aligned} f(t) &= \langle x, x \rangle + t \langle y, x \rangle + t \langle x, y \rangle + t^2 \langle y, y \rangle \\ &= \langle x, x \rangle + 2t \langle x, y \rangle + t^2 \langle y, y \rangle \end{aligned}$$

We'll find the minimum:

$$f'(t) = 2\langle x, y\rangle + 2t\langle y, y\rangle = 0$$

so

$$t_{\min} = -\frac{\langle x, y\rangle}{\langle y, y\rangle}$$

And hence,

$$\begin{aligned} f(t_{\min}) &= \langle x, x\rangle - 2\frac{\langle x, y\rangle}{\langle y, y\rangle}\langle x, y\rangle + \frac{\langle x, y\rangle^2}{\langle y, y\rangle^2}\langle y, y\rangle \\ &= \langle x, x\rangle - \frac{\langle x, y\rangle^2}{\langle y, y\rangle} \geq 0 \end{aligned}$$

from which follows the inequality. □

Proposition 2.3 Induced norm: If $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, then the function

$$\|x\| := \sqrt{\langle x, x\rangle}$$

is a norm on X. We say this is the norm induced by the inner product.

Proof. Clearly, our function $\|x\| \geq 0$ and that $\|x\| = 0 \iff \langle x, x\rangle = 0 \iff x = 0_X$.

Next,

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha^2\langle x, x\rangle} = |\alpha|\|x\|$$

and finally,

$$\begin{aligned} \|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &\leq \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle \\ &\leq \langle x, x\rangle + 2\sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle} + \langle y, y\rangle \\ &= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= \big(\|x\| + \|y\|\big)^2 \end{aligned}$$

□

Another consequence: angle between vectors Let x, y be two nonzero vectors. Then $\frac{\langle x,y\rangle}{\|x\|\|y\|} \in [-1, 1]$.

Hence, $\exists!\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|\|y\|}$$

By definition, θ is the angle between x and y.

Also, $\langle x, y\rangle = \|x\|\|y\|\cos\theta$. x and y are called orthogonal if $\theta = \pi/2 \iff \langle x, y\rangle = 0$. We write $x \perp y$.

Theorem 2.4 *Pythagorean theorem: In any inner product space, we have P.T.:*

If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Proof.

$$\|x+y\|^2 = \langle x, x\rangle + 2\underbrace{\langle x, y\rangle}_{=0} + \langle y, y\rangle = \|x\|^2 + \|y\|^2$$

□

Example 2: $X = \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$. $\|x\| = \left(\sum_{k=1}^n x_k^2\right)^{1/2}$.

Notice that $\langle x, y\rangle = \sum_{k=1}^n x_k y_k$ is an inner product on $\mathbb{R}^n$, and $\|\cdot\|$ is induced by $\langle\cdot,\cdot\rangle$, so $\|\cdot\|$ is a norm.

Also, if $x, y \in \mathbb{C}^n$, then

$$\langle x, y\rangle := \sum_{k=1}^n \overline{x_k} y_k$$

Example 5: $C([a,b])$, $\|f\| = \left(\int_a^b (f(x))^2 \,\mathrm{d}x\right)^{1/2}$ is a norm.

Check that $\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$ is an inner product.

1. $\langle f, f\rangle = \int_a^b f^2(x)\,\mathrm{d}x \geq 0$ ✓
2. $\langle f, f\rangle = \int_a^b f^2(x) = 0 \iff f^2(x) = 0 \iff f(x) = 0$ ✓
3. $\langle f, \alpha g + \beta h\rangle = \int_a^b f(x)(\alpha g(x) + \beta h(x))\,\mathrm{d}x = \alpha\int_a^b f(x)g(x)\,\mathrm{d}x + \beta\int_a^b f(x)h(x)\,\mathrm{d}x = \alpha\langle f, g\rangle + \beta\langle f, h\rangle$ ✓
4. $\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x = \int_a^b g(x)f(x)\,\mathrm{d}x = \langle g, f\rangle$ ✓

Thus, $\|f\| = \sqrt{\langle f, f\rangle}$ is a norm induced by an inner product.

This is cool! This is an infinite dimensional vector space of functions, but we still have notions of angles.

For example, $\sin x, \cos x \in C([0, 2\pi])$.

$$\langle \sin x, \cos x\rangle = \int_0^{2\pi} \sin x \cos x \,\mathrm{d}x = 0$$

i.e. $\sin x \perp \cos x$. This is the basis of Fourier analysis.

Exercise 2.5 Find the angle between $f(x) = x$ and $g(x) = x^2$ in $C([0,1])$.

Theorem 2.6 ***(Parallelogram law)*** *If $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is the induced norm. Then*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Remark. This has the geometrical interpretation of the left hand side being the sum of the length squared of the diagonals of a parallelogram formed out of x and y, and the right hand side being the length squared of the sides.

Proof. The proof follows just by expanding the left hand side:

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= 2\langle x, x\rangle + 2\langle y, y\rangle + 2\langle x, y\rangle - 2\langle x, y\rangle \\ &= 2\left(\|x\|^2 + \|y\|^2\right)\end{aligned}$$

□

§3 Friday, September 6, 2013

End of last class: In any inner product space the parallelogram law holds. In other words, if $(X, \langle\cdot,\cdot\rangle)$ is an IPS and

$$\|x\| = \sqrt{\langle x, x\rangle}$$

is induced norm on X, then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all $x, y \in X$. This result has an important converse.

Converse to parallelogram law:

Given a normed space $(X, \|\cdot\|)$, when is there an inner product on X that induces the norm $\|\cdot\|$? In other words, given the norm $\|\cdot\|$, when can we find an inner product $\langle\cdot,\cdot\rangle$ so that the norm is induced by the inner product?

$$\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle} \tag{$\star$}$$

This can be done very rarely.

Theorem 3.1 *Let $(X, \|\cdot\|)$ be a normed space. Then there exists an inner product $\langle\cdot,\cdot\rangle$ on X that induces $\|\cdot\|$, i.e. ($\star$) holds,* if and only if *the norm $\|\cdot\|$ satisfies the parallelogram law: $\forall x, y \in X$,*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

One direction is obvious. The other direction is non-trivial.

If the parallelogram law holds, then there exists an inner product that induces $\|\cdot\|$ (non-trivial).

This inner product must be

$$\langle x, y\rangle = \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

This will be proven in the first tutorial (and a proof that π is irrational).

Example 2:

$$\|x\| = \max_{1\le k\le n} |x_k|$$

where $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$.

Is this norm induced by an inner product?

The answer is "NO" if $n \ge 2$. The easiest way is to show that the parallelogram law fails for two vectors x, y.

Take $x = (1, 1, 0, \ldots, 0)$ and $y = (1, -1, 0, \ldots, 0)$. Then

$$x + y = (2, 0, \ldots, 0)$$

and

$$x - y = (0, 2, 0, \ldots, 0)$$

So that $\|x+y\| = 2$, $\|x-y\| = 2$, $\|x\| = 1$, $\|y\| = 1$. Then $\|x+y\|^2+\|x-y\|^2 = 8$ but $2(\|x\|^2+\|y\|^2) = 4$.

Exercise 3.2 $C([a, b])$,

$$\|f\| = \max_{a\le x\le b} |f(x)|$$

Show the parallelogram law fails for two functions $f, g \in C([a, b])$.

§3.1 Holder and Minkowski inequalities

If you want to show $A = B$, you often do so by showing $A \le B$ and $B \le A$.

Lemma 3.3 *Let $a, b \ge 0$ and let $p, q > 1$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

(p, q are "indicies", and if they satisfy the above relation, they are "conjugate").

Then

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$$

If $p = q = 2$, then $(a-b)^2 \ge 0 \iff a^2 + b^2 \ge 2ab \implies \frac{1}{2}a^2 + \frac{1}{2}b^2 \ge ab$.

Proof. If $a = 0$ or $b = 0$ then the inequality is trivial. Let $a > 0, b > 0$ and $\alpha, \beta \in \mathbb{R}$ be such that

$$a = e^{\frac{\alpha}{p}} \text{ and } b = e^{\frac{\beta}{q}}$$

i.e. $\alpha = p \log a$ and $\beta = q \log b$.

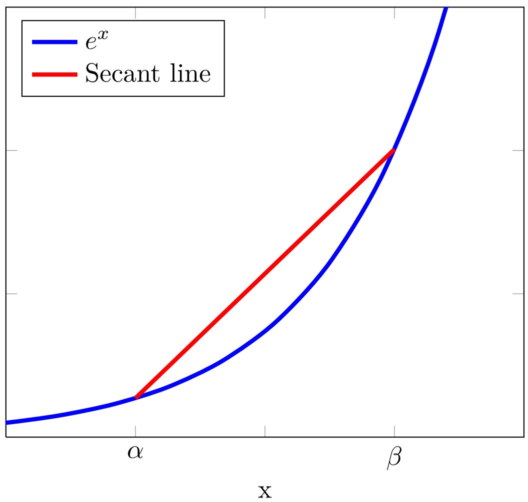
Then

$$ab = e^{\frac{\alpha}{p} + \frac{\beta}{q}}$$

Let $f(x) = e^x$, $f''(x) = e^x > 0$.

So f is convex.

Notice any secant line is above the function:



Convexity is defined as

$$f(t\alpha + (1-t)\beta) \leq tf(\alpha) + (1-t)f(\beta)$$

for all $t \in [0, 1]$.

Let $t = \frac{1}{p}$, so that $1 - t = 1 - \frac{1}{p} = \frac{1}{q}$.

Then

$$\begin{aligned} ab = e^{\frac{\alpha}{p} + \frac{\beta}{q}} &= f\left(\frac{\alpha}{p} + \frac{\beta}{q}\right) \\ &\leq \frac{1}{p} f(\alpha) + \frac{1}{q} f(\beta) = \frac{1}{p} e^\alpha + \frac{1}{q} e^\beta = \frac{1}{p} a^p + \frac{1}{q} b^q \end{aligned}$$

□

Proposition 3.4 Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and let $p, q > 1$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then

$$\sum_{k=1}^n |x_k| \cdot |y_k| \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}$$

If $p = q = 2$ then Holder Inequality follows from Cauchy Schwarz.

Proof. Let $A = \left(\sum_{k=1} |x_k|^p\right)^{1/p}$, $B = \left(\sum_{k=1} |y_k|^q\right)^{1/q}$.

If $A = 0$ or $B = 0$, the inequality is trivial.

So we can assume $A > 0, B > 0$.

$$\underbrace{\frac{|x_k|}{A}}_{a} \cdot \underbrace{\frac{|y_k|}{B}}_{b} \leq \frac{1}{p}\frac{|x_k|^p}{|A|^p} + \frac{1}{q}\frac{|y_k|^q}{B^q}$$

apply Lemma

Now we'll sum over k.

$$\sum_{k=1}^{n} \frac{|x_k| \cdot |y_k|}{AB} \leq \frac{1}{p} \underbrace{\sum_{k=1}^{n} \frac{|x_k|^p}{A^p}}_{=1} + \frac{1}{q} \underbrace{\sum_{k=1}^{n} \frac{|y_k|^q}{B^q}}_{=1} = 1$$

because

$$\frac{\sum_{k=1}^{n} |x_k|^p}{A^p} = \frac{\sum_{k=1}^{n} |x_k|^p}{\sum_{k=1}^{n} |x_k|^p} = 1$$

So,

$$\frac{\sum_{k=1}^{n} |x_k| \cdot |y_k|}{AB} \leq 1$$

which yields the Holder inequality. □

Proposition 3.5 (Minkowski Inequality:) Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, and $p \geq 1$. Then

$$\left(\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{1/p}$$

Proof. Case $p = 1$ is obvious. Assume $p > 1$.

$$\begin{aligned}\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^p &= \sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1} \left(|x_k| + |y_k|\right) \\ &= \sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1} |x_k| + \sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1} |y_k|\end{aligned}$$

We now apply Holder inequality

$$\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1} |x_k| \leq \left(\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{q(p-1)} \right)^{1/q} \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}$$

$$\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{p-1} |y_k| \leq \left(\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^{q(p-1)} \right)^{1/q} \left(\sum_{k=1}^{n} |y_k|^p \right)^{1/p}$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

So

$$\frac{1}{q} = \frac{p-1}{p}$$

and hence

$$q(p-1) = p$$

So, summing up,

$$\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^p \leq \left(\sum_{k=1}^{n} \left(|x_k| + |y_k|\right)^p \right)^{1/q} \left(\left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{1/p} \right)$$

Dividing,

$$\left(\sum_{k=1}^{n}\left(|x_k|+|y_k|\right)^p\right)^{1-\frac{1}{q}} \le \left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n}|y_k|^p\right)^{1/p}$$

Done! $\square$

§3.2 $\boxed{\ell_p\text{-norms}}$

$\mathbb{R}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\|x\|_p = \left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}$$

For $p \ge 1$, $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

$\|x\|_p \ge 0$, $\|x\|_p = 0 \iff x = 0_X$, $\|\alpha x\|_p = |\alpha|\|x\|_p$.

$$\begin{aligned}\|x+y\|_p = \left(\sum_{k=1}^{n}|x_k+y_k|^p\right)^{1/p} &\le \left(\sum_{k=1}^{n}\left(|x_k|+|y_k|\right)^p\right)^{1/p}\\ &\le \left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n}|y_k|^p\right)^{1/p}\\ &= \|x\|_p + \|y\|_p\end{aligned}$$

Consider $0 < p < 1$. then $\|\cdot\|_p$ is not a norm for $n \ge 2$.

The triangle inequality fails. Take $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$. Then $\|x\|_p = \|y\|_p = 1$ and $\|x+y\|_p^p = 2$.

But $2^{1/p} > 2$ if $0 < p < 1$, so

$$\|x+y\|_p > \|x\|_p + \|y\|_p$$

and hence $\|\cdot\|_p$ is not a norm.

§4 Tuesday, September 10, 2013

$$p \to \|x\|_p$$

assignment problem

$$\lim_{p\to\infty}\|x\|_p = \max_{1\le k\le n}|x_k| = \|x\|_\infty$$

Exercise 4.1 For $x, y \in \mathbb{R}^n$, set

$$d(x,y) = \sum_{n=1}^{n}|x_k - y_k|^p$$

for $0 < p < 1$. Show that d is a metric on $\mathbb{R}^n$.

Which of the norms $\|\cdot\|_p$ is induced by an inner product? The answer is: only $p = 2$ ($n \ge 2$). To show this, one verifies that the parallelogram law fails if $p \ne 2$.

Take $x = (1, 1, 0, \ldots, 0)$ and $y = (1, -1, 0, \ldots, 0)$. Then $\|x+y\|_p = 2$, $\|x-y\|_p = 2$, $\|x\|_p = 2^{1/p}$, $\|y\|_p = 2^{1/p}$.

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 8$$

and equality holds only

$$2(\|x\|_p^2 + \|y\|_p^2) = 4 \cdot 2^{2/p} \iff p = 2$$

§4.1 Some basic definitions in metric spaces

Definition 4.2 Take (X, d) a metric space, and $Y \subset X$. The restriction of d to $Y \times Y$ satisfies all properties of a metric. So (Y, d) is a metric space with respect to (X, d), and (Y, d) is called a metric subspace of (X, d).

Definition 4.3 (X_k, d_k), $1 \le k \le n$, metric spaces. Then

$$X = \bigtimes_{k=1}^{n} X_k = \prod_{k=1}^{n} X_k = \{x = (x_1, \dots, x_n), x_k \in X_k\}$$

with

$$d(x, y) = \left(\sum_{k=1}^{n} \big(d_k(x_k, y_k)\big)^2 \right)^{1/2}$$

(X,d) is called a product of metric spaces $(X_k, d_k),\ 1 \le k \le n$

Let $x, y, z \in X$

$$\begin{aligned} d(x, y) = \left(\sum_{k=1}^{n} \big(d_k(x_k, y_k)\big)^2 \right)^{1/2} &\le \left(\sum_{k=1}^{n} \big(d_k(x_k, z_k) + d_k(z_k, y_k)\big)^2 \right)^{1/2} \\ &\le \left(\sum_{k=1}^{n} \big(d_k(x_k, y_k\big)^2 \right)^{1/2} + \left(\sum_{k=1}^{n} d_k(y_k, z_k)^2 \right)^{1/2} \end{aligned}$$

By the Minkowski inequality.

So $d(x, y) \le d(x, z) + d(z, y)$

Definition 4.4 Two metrics d_1 and d_2 of a set X are called equivalent if $\exists C_1, C_2 > 0$ s.t. $\forall x, y \in X$

$$C_1 d_2(x, y) \le d_1(x, y) \le C_2 d_2(x, y)$$

$d_1 \sim d_2 \iff d_1, d_2$ are equivalent metrics
is a relation of equivalence on the set of metrics on X:

Transitive: $d_1 \sim d_2$ and $d_2 \sim d_3 \implies d_1 \sim d_3$ (Exercise)

Symmetric: $d_1 \sim d_2 \implies d_2 \sim d_1$

Reflexive: $d_1 \sim d_1$

If $d_1 \sim d_2$, then

$$\frac{1}{c_2} d_1(x, y) \le d_2(x, y) \le \frac{1}{c_1} d_1(x, y)$$

So $d_2 \sim d_1$.

If X is a vector space and $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X, then these two norms are called equivalent if $\exists c_1, c_2 > 0$ such that

$$c_1 \|x\|_2 \le \|x\|_1 \le c_2 \|x\|_2$$

for all $x \in X$.

If two norms are equivalent, then the induced metric are equivalent.

Fact. If $\dim X < \infty$, *then then any two norms are equivalent. This is* not true *for infinite dimensional metric spaces.*

Example 4.5 $\mathbb{R}^n, x = (x_1, ..., x_n) \in \mathbb{R}^n$.

$$|x|_1 = \sum_{k=1}^{n} |x_k|$$

$$\|x\|_\infty = \max_{1 \le k \le n} |x_k|$$

$$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$$

Back to product spaces. Equivalent metrics.

$$d_\infty(x, y) = \max_{1 \le k \le n} d_k(x_k, y_k)$$

d_∞ is equivalent to the product metric d we introduced before.

Definition 4.6 Consider a set X, a function

$$\rho : X \times X \to [0, \infty)$$

That satisfies:

1. $\rho(x, y) \ge 0$
2. If $x = y \Rightarrow \rho(x, y) = 0$
3. $\rho(x, y) = \rho(y, x)$
4. $\rho(x, y) \le \rho(x, z) + \rho(z, y)$

Then ρ is called a pseudo-metric on X, and (X, ρ) is a pseudo-metric space.

The difference between a metric and a pseudo-metric is that for ρ it is possible that

$$x \ne y, \text{ but } \rho(x, y) = 0$$

In many constructions in analysis, one arrives first at a pseudo-metric space. There is a natural and canonical way to construct a metric space out of a pseudo-metric space.

We introduce a relation $\sim$ on X by $x \sim y \iff \rho(x, y) = 0$.

This is a relation of equivalence: $x \sim x, x \sim y \implies y \sim x$, and $x \sim y$, and $y \sim z \iff \rho(x, y) = 0$ and $\rho(y, z) = 0$. This implies that $0 = \rho(x, y) + \rho(y, z) \ge \rho(x, z) \ge 0 \implies \rho(x, z) = 0$. That is, $x \sim z$.

$X/\sim$ classes of equivalence. $[x]$ class of equivalence that contains x. Set

$$d([x], [y]) = \rho(x, y)$$

We have to check that d is well defined.

We have to check that if

$$[x] = [x'] \text{ and } [y] = [y']$$

Then $d([x], [y]) = d([x'], [y'])$

Suppose that $x \sim x'$ $([x] = [x'])$ and $y \sim y'$ $([y] = [y'])$

$$\rho(x, y) \le \underbrace{\rho(x, x')}_{=0} + \rho(x', y') + \underbrace{\rho(y', y)}_{=0} = \rho(x', y')$$

But we can do the same thing for $\rho(x', y')$:

$$\rho(x', y') \le \rho(x', x) + \rho(x, y) + \rho(y, y') = \rho(x, y)$$

So $\rho(x, y) = \rho(x', y')$, and d is well-defined.

Now we have to check that d is a metric:

1. $d([x],[y]) \geq 0$
2. $d([x],[y]) = 0 \iff \rho(x,y) = 0 \iff x \sim y \iff [x] = [y]$
3. $d([x],[y]) = \rho(x,y) = \rho(y,x) = d([y],[x])$
4. If $[x],[y],[z]$ are in $X/\sim$,

$$d([x],[y]) = \rho(x,y) \leq \rho(x,z) + \rho(z,y) = d([x],[z]) + d([z],[y])$$

So $(X/\sim, d)$ is a metric space.

Definition 4.7

$$(X_k, d_k),\ k = 1, 2, \ldots$$

a countable sequence of metric spaces. We have the Cartesian product

$$X = \bigtimes_{k=1}^{\infty} X_k = \prod_{k=1}^{\infty} X_k = \{x = (x_k)_{k=1}^{\infty}, x_k \in X_k\}$$

and we define

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \underbrace{\frac{d_k(x_k,y_k)}{1 + d_k(x_k,y_k)}}_{\leq 1}$$

Then (X,d) is a countable product of metric spaces.

Note. $d(x,y) < \infty \quad \forall x,y$

Proposition 4.8 d is a metric on X:

1. $d(x,y) \geq 0$

(to be continued)

§5 Tuesday, September 10, 2013: Tutorial I

§5.1 $\boxed{\pi \text{ is irrational}}$

Define π as the smallest positive zero of the sine function

$$\sin(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Theorem 5.1 *π is irrational*

Proof: Niven, 1947. Assume $\pi \in \mathbb{Q}$, i.e. $\pi = \frac{a}{b}$ for some $a, b \in \mathbb{N}$ (we already defined π to be positive).

Define

$$f(x) := \frac{x^n(a-bx)^n}{n!}$$

for some fixed $n \in \mathbb{N}$.

Observation: $f(x) \geq 0$, for $x \in [0, \pi]$.
Define

$$F(x) = f(x) - f^{(2)}(x) + \ldots + (-1)^n f^{2n}(x) \qquad = \sum_{k=0}^{n} (-1)^k f^{2k}(x)$$

Claim. $F(0) + F(\pi)$ *is an integer*

Proof.

$$\begin{aligned} f(x) &= \frac{x^n (a-bx)^n}{n!} \\ &= \sum_{k=0}^{2n} \frac{c_k}{n!} x^k \end{aligned}$$

with $c_k = 0$ for all $k < n$

$$\begin{aligned} \implies f^k(0) &= 0 \quad \forall k < n \\ f^k(0) &= \frac{k!}{n!} c_k \quad \forall k \geq n \end{aligned}$$

and hence, $f^k(0) \in \mathbb{Z}$ for all k, and so $F(0)$ is an integer.
(Exercise): Show $F(\pi) = 0$ using that $F(\pi - x) = f(x)$ (prove this too).
With this, the claim is proven. □

Claim. $\int_0^\pi f(x) \sin(x)\, \mathrm{d}x = F(0) + F(\pi)$

Proof.

$$F''(x) = \sum_{k=0}^{n} (-1)^k f^{2k+2}(x)$$

$$F''(x) + F(x) = \sum_{k=0}^{n} (-1)^k [f^{2k}(x) + f^{2k+2}(x)] = f(x) + (-1)^n \underbrace{f^{2n+2}(x)}_{=0}$$

Substituting into the integral,

$$\begin{aligned} \int_0^\pi f(x) \sin(x)\, \mathrm{d}x &= \int_0^\pi F''(x) \sin x\, \mathrm{d}x + \int_0^\pi F(x) \sin x\, \mathrm{d}x \\ &= F'(x) \sin x|_0^\pi - \int_0^\pi F'(x) \cos x\, \mathrm{d}x + -\cos x F(x) + \int_0^\pi F'(x) \cos(x)\, \mathrm{d}x \end{aligned}$$

The integrals cancel, and $\sin(\pi) = 0$ by definition of π, so we are left with

$$= F(\pi) \cos(\pi) + F(0) \cos(0) = F(0) + F(\pi)$$

□

Note that we also know that both $f(x), \sin x > 0$ in the the interior of the interval, and so $F(0) + F(\pi) > 0$, i.e. $F(0) + F(\pi) \geq 1$.

Now define $g(x) = x(a - bx)$. This is a parabola facing down, with zeros at 0 and π. Also note that $\frac{g(x)^n}{n!} = f(x)$.

We know that $\forall x \in [0, \pi]$, $g(x) \geq 0$, and the maximum occurs at $\frac{\pi}{2}$, and $g(\pi/2) = \frac{\pi a}{4} < \pi a$.

This implies that

$$0 \leq g(x) \leq \pi a \text{ for } x \in [0, \pi]$$

We also know that

$$0 \leq \sin x \leq 1 \text{ for } x \in [0, \pi]$$

Thus,

$$1 \leq F(0) + F(\pi) = \int_0^\pi g(x) \sin x \, \mathrm{d}x \leq \int_0^\pi \frac{(\pi a)^n}{n!} \cdot 1 \, \mathrm{d}x = \frac{\pi(\pi a)^n}{n!} < 1 \text{ for } n \text{ large enough}$$

□

§5.2 The converse to the parallelogram law

Exercise 5.2 Complexify this next proof.

Theorem 5.3 *Let $(V, \|\cdot\|)$ be a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$). T.F.A.E.*

1. *$\exists \langle \cdot, \cdot \rangle$ an inner product on V such that $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$*
2. *The parallelogram law*
$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$
holds for all $x, y \in V$.

Proof. (1) $\implies$ (2) was done in class.

So assume (2) and show (1): We will assume V is over $\mathbb{R}$. Set

$$\langle x, y \rangle = \tfrac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 \right)$$

1) $\langle x, y \rangle = \langle y, x \rangle$: obvious, exercise

2) Linearity: $\forall \alpha, \beta \in \mathbb{R}$ and $x, y, z \in V$, $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$: First we will show that for $x, y, z \in V$ that $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

By the parallelogram law,

$$2\|x+z\|^2 + 2\|y\|^2 = \|x+y+z\|^2 + \|x-y+z\|^2$$

So

$$\implies \|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2$$

but also, by another parallelogram law,

$$= 2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2$$

combining the two

$$= \|x+z\|^2 + \|y+z\|^2 + \|x\|^2 + \|y\|^2 - \tfrac{1}{2}\|x-y+z\|^2 - \tfrac{1}{2}\|y-x+z\|^2$$

now doing $z \to -z$,

$$\|x+y-z\|^2 = \|x-z\|^2 + \|y-z\|^2 + \|x\|^2 + \|y\|^2 - \tfrac{1}{2}\|y-x+z\|^2 - \tfrac{1}{2}\|x-y+z\|^2$$

Now,

$$\langle x+y, z \rangle = \tfrac{1}{4} \left(\|x+y+z\|^2 - \|x+y-z\|^2 \right)$$

so by using the relations above,

$$\begin{aligned}
&= \tfrac{1}{4} \left(\|x+z\|^2 + \|y+z\|^2 - \|y-z\|^2 \right) \\
&= \tfrac{1}{4} \left(\|x+z\|^2 - \|x-z\|^2 \right) + \tfrac{1}{4} \left(\|y+z\|^2 - \|y-z\|^2 \right) \\
&= \langle x, z \rangle + \langle y, z \rangle
\end{aligned}$$

So we have $\forall x, y, z \in V$, we have

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

And so for $n \in \mathbb{N}$,

$$\langle nx, y\rangle = n\langle x, y\rangle$$

by applying this additivity property many times.

Also,

$$\langle 0, y\rangle = \langle 0+0, y\rangle = \langle 0, y\rangle + \langle 0, y\rangle$$

and hence, $\langle 0, y\rangle = 0$.

Additionally,

$$0 = \langle 0, y\rangle = \langle x-x, y\rangle = \langle x, y\rangle + \langle -x, y\rangle \implies \langle -x, y\rangle = -\langle x, y\rangle$$

so for $a \in \mathbb{Z}$, $\langle ax, y\rangle = a\langle x, y\rangle$.

Next,

$$\begin{aligned} \langle x, y\rangle &= \langle n\frac{x}{n}, y\rangle = n\langle \frac{x}{n}, y\rangle \\ \implies \langle \frac{x}{n}, y\rangle &= \frac{1}{n}\langle x, y\rangle \end{aligned}$$

So $\forall \lambda \in \mathbb{Q}$,

$$\langle \lambda x, y\rangle = \lambda xy$$

by combining the fact for integers with denominators.

Fact. As defined, the function $f_y : V \to \mathbb{R}$ given by $f_y(x) = \langle x, y\rangle$ is continuous.

(exercise)

Because the rationals are dense in the reals, and the inner product is continuous, we have

$$\forall \lambda \in \mathbb{R}, \langle \lambda x, y\rangle = \lambda\langle x, y\rangle$$

$\square$

Hint for the complex case: Define

$$\langle x, y\rangle = \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) + \tfrac{1}{4}i\left(\|x+iy\|^2 - \|x-iy\|^2\right) = \tfrac{1}{4}\sum_{k=0}^{3} i^k\|x+i^k y\|^2$$

§6 Thursday, September 12, 2013

Office hours: Friday's 11:30-13:30, Burnside 1018.

Last class: products of countably many metric spaces.

$$(X_k, d_k), k = 1\,2\,\ldots$$

metric spaces.

$$X = \prod_{k=1}^{\infty} X_k = \{x = (x_k)_{k=1}^{\infty} : x_k \in X_k\}$$

with

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k}\underbrace{\frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}}_{\leq 1}$$

Note. $d(x, y) < \infty \quad \forall x, y$

Proposition 6.1 d is a metric on X

Proof.

1. $d(x,y) \geq 0$ clear
2. $d(x,y) = 0 \iff x_k = y_k, \forall k \iff x = y$
3. $d(x,y) = d(y,x)$

   Fact. If $a, b, c \geq 0$ such that
   $$a \leq b + c$$
   Then
   $$\frac{a}{a+1} \leq \frac{b}{1+b} + \frac{c}{1+c}$$
   Proof:
   $$\begin{aligned}\frac{a}{a+1} \leq \frac{b}{1+b} + \frac{c}{1+c} &\iff a(1+b)(1+c) \leq b(1+a)(1+c) + c(1+a)(1+b)\\ &\iff a(1+b+c+bc) \leq b(1+a+c+ac) + c(1+a+b+ab)\\ &\iff a \leq b + c + 2bc + cab\end{aligned}$$
   which is true. Hence,
4. If $x, y, z \in X$, then $d_k(x_k, y_k) \leq d(x_k, z_k) + d_k(z_k, y_k)$, and so
   $$\frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)} \leq \frac{d_k(x_k, z_k)}{1 + d_k(x_k, z_k)} + \frac{d_k(z_k, y_k)}{1 + d_k(z_k, y_k)}$$
   and so we can multiply both sides by 2^{-k} and sum up.

□

§6.1 Ch. 9.2: Open sets, closed sets, Convergence

Let (X, d) be a metric space.

Definition 6.2 Let $x \in X$ and $r > 0$. The set $B(x, r) = \{y \in X : d(x, y) < r\}$. Then $B(x, r)$ is called the open ball centered at a point x of radius $r > 0$.

Definition 6.3 A set $O \subset X$ is called open if for any $x \in O$, there exists an $r_x > 0$ such that $B(x, r_x) \subset O$. Any open set that contains x is called a neighborhood of x.

Claim. Any open ball is an open set.

Proof. Consider $B(x, r)$. Let $x' \in B(x, r)$. We need to show that there exists an $r_{x'} > 0$ such that $B(x', r_{x'}) \subset B(x, r)$.

Take $r_{x'} = r - d(x, x') > 0$. Let $y \in B(x', r_{x'})$, then $d(x, y) \leq d(x, x') + d(x', y) < d(x, x') + r_{x'} = r$ So $d(x, y) < r$, and so $y \in B(x, r)$. □

Note. "Geometry" of open balls depends on metric.

$\mathbb{R}^2, \|x\| = \sqrt{x_1^2 + x_2^2}$. Then $B(0, 1)$ is the unit disc.

Or $\|x\| = \max(|x_1|, |x_2|)$. Then $B(0, 1) = \{x \in \mathbb{R}^2 : \max(|x_1|, |x_2|) \leq 1\}$, i.e. $B(0, 1)$ is the square with sides of length 1 centered at the origin.

Consider the discrete metric
$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$
on a set X. Then
$$B(x, r) = \{y \in X : d(x, y) < r\}$$
If $r < 1$, $B(x, r) = \{x\}$. If $r > 1$, $B(x, r) = X$.

Note. X itself is an open set. The empty set is also an open set by definition.

Proposition 6.4

1. If $\{O\}_{i\in I}$ is a family of open sets in (X,d) then
$$\bigcup_{i\in I} O_i$$
is an open set.

2. If $O_1,\ldots,O_n$ are open sets (finitely many), then
$$O_1\cap\ldots\cap O_n$$
is also open.

Proof. Let $x\in\bigcup_{i\in I} O_i$. Then $x\in O_{i_0}$, for some $i_0\in I$ and since O_{i_0} is open, $\exists r_x>0$ such that $B(x,r_x)\subset O_{i_0}$. and so
$$B(x,r_x)\in\bigcup_{i\in I} O_i$$

To prove (2), let
$$x\in O_1\cap\ldots\cap O_n$$
Then $x\in O_k$ for $1\le k\le n$ and because O_k is open, $\exists r_x^{(k)}>0$ such that
$$B(x,r_x^{(k)})\subset O_k$$
Take
$$r_x=\min_{1\le k\le n} r_x^{(k)}$$
Then $r_x>0$, and for any k,
$$B(x,r_x)\subset B(x,r_x^{(k)})\subset O_k$$
and hence
$$B(x,r_x)\subset\bigcap_{k=1}^{n} O_k$$

Remark. We require only finitely many O_k for the intersection case, because otherwise the $r_x^{(k)}$'s infimum may be 0 even though each one is positive, and so the proof does not hold. In fact, it is not true: take $\mathbb{R}$ with $d(x,y)=|x-y|$, and $B(0,\frac{1}{k})=(-\frac{1}{k},\frac{1}{k})$. The intersection of such balls for all k is just the set $\{0\}$, which is not open: for any $r>0$, $B(0,r)=(-r,r)$ is not contained in $\{0\}$.

Proposition 6.5 TFAE:

1. O is an open set in (X,d)
2. $O=\bigcup_{i\in I} B(x_i,r_i)$

Proof. (2) $\implies$ (1) : because of last proposition and the fact that $B(x_i,r_i)$ is open.

To prove (1) $\implies$ (2), note that for every $x\in O$, $\exists r_x>0$ such that $B(x,r_x)\subset O$.

Then
$$O\subset\bigcup_{x\in O} B(x,r_x)\subset O$$
And hence
$$O=\bigcup_{x\in O} B(x,r_x)$$

Proposition 6.6 Let (X,d) be a metric space (Y,d) a metric subspace. Then a set $U\subset Y$ is open in (Y,d) if and only if
$$U=V\cap Y$$
where V is an open set in (X,d).

Proof. $B_Y(x,r)$ denotes an open ball in Y $B_X(x,r)$ denotes an open ball in X.

Then

$$B_Y(x,r) = \{y \in Y : d(x,y) < r\} = \{y \in X : d(x,y) < r\} \bigcap Y = B_X(x,r) \bigcap Y$$

So $B_Y(x,r) = B_X(x,r) \cap Y$ for $x \in Y$.

Let $U \subset Y$ be an open set. Then by the previous proposition,

$$U = \bigcup_{i \in I} B_Y(x,r) = \bigcup_{i \in I} (B_X(x_i,r_i)) \bigcap Y = \underbrace{\left(\bigcup_{i \in I} B_X(x_i,r_i) \right)}_{V} \bigcap Y$$

where $V = \left(\bigcup_{i \in I} B_X(x_i,r_i) \right)$ is an open set in X.

To prove the opposite inclusion, let V be open in X.

Then

$$V = \bigcup_{i \in I} B_X(x_i,r_i)$$

and

$$V \bigcap Y = \bigcup_{i \in I} B_X(x_i,r_i) \bigcap Y$$

If $x_i \in Y$ then $B_X(x_i,r_i) \bigcap Y = B_Y(x_i,r_i)$ and is open in Y.

If $x_i \notin Y$ it is left as an exercise to prove that $B_X(x_i,r_i) \bigcap Y$ is either empty or open in Y so $V \bigcap Y$ is union of open sets and hence open. □

§6.2 Closed sets

Let (X,d) be a metric space.

Definition 6.7 Let $A \subset X$. A point $x \in X$ is called a closure point of A if every neighborhood of x contains a point of A.

The collection of all closure points of A is denoted by $\bar{A}$ or $\text{cl}(A)$ and is called the closure of A.

x is a closure point $\iff$ for any open set O that contains x, $O \cap A \neq \varnothing = \{\}$.

Note that $A \subset \text{cl}(A)$ as $x \in A$, $x \in O$, means $\{x\} \in O \cap A$.

Definition 6.8 A set A is called closed if $\text{cl}(A) = A$.

§7 Friday, September 13, 2013

Proposition 7.1 $\text{cl}(A)$ is a closed set. Moreover, $\text{cl}(A)$ is the smallest closed set that contains A. In other words, if F is any other closed set that contains A, $\text{cl}(A) \subseteq F$.

Proof. To prove that $\text{cl}(A)$ is closed, we need to show that

$$\text{cl}(\text{cl}(A)) = \text{cl}(A)$$

We already know that

$$\text{cl}(A) \subseteq \text{cl}(\text{cl}(A))$$

So we need to prove that $\text{cl}(\text{cl}(A)) \subseteq \text{cl}(A)$.

Let $x \in \text{cl}(\text{cl}(A))$ and let O be a neighborhood of x. Then we know $O \cap \text{cl}(A)$ is non-empty.

Let $x' \in O \cap \text{cl}(A)$. Then O is a neighborhood of x' and x' is a closure point of A. So

$$O \cap A \neq \varnothing$$

Then $x \in \text{cl}(A)$. Our inclusion holds.

To prove the second statement, let F be any closed set that contains A. Let $x \in \text{cl}(A)$ and let O be a neighborhood of x.

Then

$$O \cap A \neq \varnothing$$

Since $A \subset F$, $O \cap F \neq \varnothing$.

This is true for any neighborhood O of x, and so $x \in \mathrm{cl}(F)$. Since F is closed, $\mathrm{cl}(F) = F$. So $x \in F$. This proves that

$$\mathrm{cl}(A) \subset F$$

□

Exercise 7.2 In the proof of the second statement of the last proposition, we have effectively shown that

$$A \subset B \implies \mathrm{cl}(A) \subset \mathrm{cl}(B)$$

Write down the details.

Proposition 7.3 A subset $A \subset X$ is closed $\iff$ its complement A^c is open.

Proof. Suppose that A is closed, that is, $A = \mathrm{cl}(A)$. We want to show that A^c is open. Let $x \in A^c$. Then $x \notin A$ and since $A = \mathrm{cl}(A)$, $x \notin \mathrm{cl}(A)$ i.e. x is not a closure point of A.

It is not true that every neighborhood of x has a nonempty intersection with A, i.e. $\exists$ a neighborhood O of x such that $O \cap A = \varnothing$, i.e. $O \subset A^c$. Then there exists an open ball $B(x, r_x)$ with $r_x > 0$ such that $B(x, r_x) \subset O \subset A^c$. This is true for any $x \in A^c$, so A^c is open.

We now prove the converse statement: suppose that A^c is open. Let $x \in \mathrm{cl}(A)$. Then for any $r > 0$, $B(x, r) \cap A \neq \varnothing$. So $B(x, r) \not\subset A^c$. Since A^c is open, we cannot have that $x \in A^c$, so $x \in A$. Hence, $\mathrm{cl}(A) \subset A$, then $A = \mathrm{cl}(A)$, i.e. A is closed. □

Proposition 7.4

1. If $\{A_i\}$ is a family of closed sets in a metric space (X, d) then

$$\bigcap_{i \in I} A_i \text{ is a closed set}$$

2. If $A_1, \ldots, A_n$ are closed, then

$$A_1 \cup A_2 \cup \ldots \cup A_n \text{ is closed}$$

Proof. (1) Since A_i is closed, A_i^c is open. Then $\bigcup_{i \in I} A_i^c$ is open by previous proposition.

Then, $\left(\bigcup_{i \in I} A_i^c\right)^c$ is closed.

$$\left(\bigcup_{i \in I} A_i^c\right)^c = \bigcap_{i \in I} (A_i^c)^c = \bigcap_{i \in I} A_i$$

is closed.

Similarly for the other part. □

§7.1 Interior and boundary of a set

$\mathrm{cl}(A)$ is the smallest closed set that contains A. Hence

$$\mathrm{cl}(A) = \bigcap_{F, A \subset F, F \text{ closed}} F$$

Definition 7.5 The interior of a set A, denoted $\mathrm{interior}(A)$, is the largest open set contained in A.

$$\mathrm{interior}(A) = \bigcup_{O, O \subset A, O \text{ open}} O$$

$\mathrm{interior}(A)$ could be $\varnothing$, but always exists.

Definition 7.6 The boundary of A, ∂A is

$$\partial A = \{x \in X : \text{for every neighborhood } O \text{ of } x,\ O \cap A \neq \varnothing,\ O \cap A^c \neq \varnothing\} = \mathrm{cl}(A) \cap \mathrm{cl}(A^c)$$

∂A is a closed set as it's the intersection of two closed sets.

$$\partial A \cap \text{interior}(A) = \varnothing$$

$x \in \text{interior}(A) \implies \exists r_x > 0$ such that $B(x, r_x) \subset \text{interior}(A) \subset A$. Then $B(x, r_x) \cap A^c = \varnothing$, so $x \notin \partial A$.

Proposition 7.7 $\mathrm{cl}(A) = \text{interior}(A) \cup \partial A$

Exercise: (Midterm problem in 2007) Let (X, d) be a metric space, $A, B \subset X$. Prove the following:

1. $\partial(A \cup B) \subset \partial A \cup \partial B$
2. Find an example of X, A, B such that
$$\partial(A \cup B) \neq \partial A \cup \partial B$$
3. Find an example of $\{A_n\}_{n=1,2,\ldots}^{\infty}$ and X such that
$$\bigcup_{n=1}^{\infty} \partial A_n \subsetneq \partial\left(\bigcup_{n=1}^{\infty} A_n\right)$$

Solution:

1. $$\partial(A \cup B) = \mathrm{cl}(A \cup B) \cap \mathrm{cl}((A \cup B)^c) = (\mathrm{cl}(A) \cup \mathrm{cl}(B)) \cap (\mathrm{cl}(A^c \cap B^c))$$
using homework problem that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
$$= (\mathrm{cl}(A) \cap \mathrm{cl}(A^c \cap B^c)) \cup (\mathrm{cl}(B) \cap \mathrm{cl}(A^c \cap B^c))$$
using an exercise that $S \subset D \implies \mathrm{cl}(S) \subset \mathrm{cl}(D)$
$$\subset (\mathrm{cl}(A) \cap \mathrm{cl}(A^c)) \cup (\mathrm{cl}(B) \cap \mathrm{cl}(B^c)) = \partial A \cup \partial B$$
2. Just take $A = (0, 1)$, and $B = (\frac{1}{2}, 2)$, where $X = \mathbb{R}$, and $d(x, y) = |x - y|$.
Then $\partial A = \{0, 1\}$ and $\partial B = \{\frac{1}{2}, 2\}$. And $A \cup B = (0, 2)$, so $\partial(A \cup B) = \{0, 2\} \neq \{0, 1, \frac{1}{2}, 2\}$.
3. Take $X = \mathbb{R}$, $d(x, y) = |x - y|$. Consider $\{r_n\}_{n=1}^{\infty}$ enumeration of $\mathbb{Q}$. Let $A_n = \{r_n\}$. Then $\partial A_n = \{r_n\}$.
$$\bigcup_{n=1}^{\infty} A_n = \mathbb{Q}$$
$\partial\mathbb{Q}$ in $\mathbb{R}$ is all of $\mathbb{R}$. Let $r \in \mathbb{R}$ be any real, and $B(r, \varepsilon) = (r - \varepsilon, r + \varepsilon)$ contains a rational and an irrational number, so $B(r, \varepsilon) \cap \mathbb{Q} \neq \varnothing$ and $B(r, \varepsilon) \cap \mathbb{Q}^c \neq \varnothing$, so $r \in \partial Q$; hence, $\mathbb{R} \subset \partial\mathbb{Q}$ and of course $\partial\mathbb{Q} \subset \mathbb{R}$, so $\partial\mathbb{Q} = \mathbb{R}$.

§7.2 Convergence in metric spaces

Definition 7.8 A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to converge to a point $x \in X$ if

$$\lim_{n\to\infty} d(x, x_n) = 0$$

or in other words,

$$\forall \varepsilon > 0, \exists N > 0, \forall n \geq N, d(x, x_n) < \varepsilon$$

x is called the limit of the sequence (x_n) and one writes

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty$$

Fact. The limit is unique: if $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} x_n = y$*, then,* $\forall n$*,*

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0$$

So $d(x, y) = 0$ *and so* $x = y$*.*

Proposition 7.9 $x \in \mathrm{cl}(A)$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$, $x_n \in A$, such that

$$\lim_{n\to\infty} x_n = x$$

Proof. Suppose $x \in \mathrm{cl}(A)$. Then, for any $n > 0$,

$$B(x, \tfrac{1}{n}) \cap A \neq \varnothing$$

Let $x_n \in B(x, \frac{1}{n}) \cap A$. x_n sequence in A and

$$d(x, x_n) \leq \tfrac{1}{n} \to 0 \text{ as } n \to \infty$$

$$\lim_{n\to\infty} x_n = x$$

□

§8 Tuesday, September 17, 2013

§8.1 Convergence in metric spaces

(X, d), $(x_n)_{n=1}^{\infty}$, $\lim_{n\to\infty} x_n = x$ iff $\lim_{n\to\infty} d(x, x_n) = 0$, x - limit point of the sequence (the limit is unique).

$$\lim_{n\to\infty} x_n = x \iff (\forall \varepsilon > 0)(\exists N > 0)(\forall n \geq N) d(x, x_n) < \varepsilon$$

Remark. A sequence (x_n) in a metric space is called bounded if for some y, $|d(y, x_n)| < M$, for some $M > 0$. This does not depend on y, since

$$d(z, x_n) \leq d(z, y) + d(y, x_n)$$

so if $d(z, x_n)$ is bounded for one y, its bounded $\forall y$.

Proposition 8.1 Convergent sequences are bounded.

Proof. We know

$$\lim_{n\to\infty} d(x, x_n) = 0$$

Take $\varepsilon = 1$. Let N be such that $\forall n \geq N$, $d(x, x_n) < 1$. Let $M = \max\{d(x, x_1), \ldots, d(x, x_N)\}$.

□

Proposition 8.2 $x \in \mathrm{cl}(A)$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in A such that $\lim_{n\to\infty} x_n = x$.

Proof. $\to$ proved last class.

$\leftarrow$ Let $x \in X$ and suppose that there exists a sequence (x_n) in A such that $\lim_{n\to\infty} x_n = x$. We want to show that $x \in \mathrm{cl}(A)$.

Let O be a neighborhood of x. Let $\varepsilon > 0$ be such that $B(x,\varepsilon) \subset O$. Since $x_n \to x$, $\exists N > 0$ such that for $n \geq N$, $x_n \in B(x,\varepsilon)$. Then $O \cap A$ is nonempty since $x_n \in A$, and x_n, for $n \geq N$, is in $B(x,\varepsilon) \subset O$.

□

Two metric spaces d_1, d_2 in set X are equivalent if $\exists C_1, C_2 > 0$ such that for all $x, y \in X$,

$$C_1 d_2(x,y) \leq d_1(x,y) \leq C_2 d_2(x,y)$$

So sequence (x_n) converges to x in (X, d_1) if and only if (x_n) converges to x in (X, d_2).
So, if d_1 and d_2 are equivalent metrics, then (X,d_1) and (X,d_2) have the same open/closed sets.
If $A \subset X$, then $\mathrm{cl}(A)_{d_1}$ denotes the closure of A in (X, d_1).

$$\begin{aligned} x \in \mathrm{cl}(A)_{d_1} &\iff \exists (x_n) \in A, \lim_{n\to\infty} d_1(x_n, x) = 0 \\ &\iff \exists (x_n) \in A, \lim_{n\to\infty} d_2(x_n, x) = 0 \iff x \in \mathrm{cl}(A)_{d_2} \end{aligned}$$

So if A is closed in (X, d_1), then $A = \mathrm{cl}(A)_{d_1} = \mathrm{cl}(A)_{d_2}$ so A is closed in (X, d_2).
Exercise: show this directly.

§8.2 Convergence in product spaces

Let (X_k, d_k) $k = 1, 2, \ldots$ be a sequence of metric spaces,

$$X = \prod_{k=1}^{\infty} X_k = \{x = (x_k)_{k=1}^{\infty} : x_k \in X_k\}$$

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

Let $x^{(n)} = (x_k^{(n)})_{k=1}^{\infty}$, $n = 1, 2, \ldots$ be a sequence in X. $x^{(n)} \to x = (x_k)_{k=1}^{\infty}$ if and only if each component of the sequence $x^{(n)}$ converges to the corresponding component of the sequence x.

In other words,

$$\lim_{n\to\infty} d(x^{(n)}, x) = 0 \iff \forall k, \lim_{n\to\infty} d_k(x_k^{(n)}, x_k) = 0$$

Proof. Suppose first that $\forall k$, $\lim_{k\to\infty} d_k(x_k^{(n)}, x_k) = 0$.

Let $\varepsilon > 0$ be given. Choose K_0 such that

$$\sum_{k \geq K_0} 2^{-k} < \frac{\varepsilon}{2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{K_0 - 1} 2^{-k} \frac{d_k(x_k^{(n)}, x_k)}{1 + d_k(x_k^{(n)}, x_k)} = 0$$

(to the n?)
So, $\exists N_0 > 0$ such that $\forall n \geq N_0$,

$$\sum_{k=1}^{K_0 - 1} 2^{-k} \frac{d_k(x_k^{(n)}, x_k)}{1 + d_k(x_k^{(n)}, x_k)} < \frac{\varepsilon}{2}$$

Therefore, for $n \geq N_0$,

$$d(x^{(n)}, x) = \sum_{k=1}^{K_0 - 1} 2^{-k} \frac{d_k(x_k^{(n)}, x_k)}{1 + d_k(x_k^{(n)}, x_k)} + \sum_{k=K_0}^{\infty} 2^{-k} \frac{d_k(x_k^{(n)}, x_k)}{1 + d_k(x_k^{(n)}, x_k)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

So $x^{(n)} \to x$ in (X, d).

To prove the converse, suppose that $\lim_{n\to\infty} d(x, x^{(n)}) = 0$.

Then, since

$$2^{-k} \frac{d_k(x_k^{(n)}, x_k)}{1 + d_k(x_k^{(n)}, x_k)} \leq d(x^{(n)}, x) \forall k$$

We have that

$$\lim_{n\to\infty} \frac{d_k(x_k^{(n)}, x_k)}{1 + d_k(x_k^{(n)}, x_k)} = 0$$

$$\lim_{n\to\infty} \left(1 - \frac{d_k(x_k^{(n)}, x_k)}{1 + d_k(x_k^{(n)}, x_k)}\right) = 1$$

$$\lim_{n\to\infty} \frac{1}{1 + d_k(x_k^{(n)}, x_k)} = 1 \implies \lim_{n\to\infty} 1 + d_k(x_k^{(n)}, x_k) = 1 \implies \lim_{n\to\infty} d_k(x_k^{(n)}, x_k) = 0$$

□

§8.3 Remarks on ℓ^p and L^p spaces

$\mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$, $\|x\|_\infty = \max_{1\leq k\leq n} |x_k|$. For $p \in [1, \infty]$, $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$, $(\mathbb{R}^n, \|\cdot\|_p)$ is denoted $\ell^p(\mathbb{R}^n)$. The norms $\|\cdot\|_p$ are equivlaent,

$$\|\cdot\|_\infty \leq \|\cdot\|_1 \leq n\|\cdot\|_1$$

and this is true in any finite dimensional vector space X. Any two norms on X are equivalent.

$\ell^p(\mathbb{N})$ is the vector space of all sequences of real numbers $x = (x_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty |x_n|^p < \infty$$

$$\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p}$$

Then $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$.

So $\ell^\infty(\mathbb{N}) = ?$ space and all bounded sequences.

(x_n), $\|x\|_\infty = \sup |x_n|$, $\|\cdot\|$ is a norm on $\ell^\infty(\mathbb{N})$.

$C([a, b])$ vector space of all continuous functions $f : [a, b] \to \mathbb{R}$.

$$\|f\|_p = \left(\int_a^b |f(x)|^p\right)^{1/p}$$

for $p > 0$. We have that $\|f\|_p \geq 0$, $\|f\|_p = 0 \iff f(x) = 0$, $\|\alpha f\|_p = |\alpha|\|f\|_p$.

If $1 < p$, $q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b |f(x)||g(x)| \,\mathrm{d}x \leq \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p} \left(\int_a^b |g(x)|^q \,\mathrm{d}x\right)^{1/q}$$

$$\|fg\|_1 \leq \|f\|_p\|g\|_q$$

Proof.

$$A = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}$$

$$B = \left(\int_a^b |g(x)|^q \, \mathrm{d}x \right)^{1/q}$$

WTS:

$$\int_a^b |f(x)|g(x)| \, \mathrm{d}x \leq AB$$

If $A = 0$, $f = 0$, inequality trivial. If $B = 0$ similarly.
Assume $A > 0$, $B > 0$.

$$\frac{|f(x)|}{A} \frac{|g(x)|}{B} \leq \frac{1}{p} a^p + \frac{1}{q} b^q = \frac{1}{p} \frac{|f(x)|^p}{A^P} + \frac{1}{q} \frac{|g(x)|^q}{B^q}$$

$$\frac{1}{AB} \int_a^b |f(x)||g(x)| \, \mathrm{d}x \leq \frac{1}{p} \frac{1}{A^p} \int_a^b |f(x)|^p \, \mathrm{d}x + \frac{1}{q} \frac{1}{B^q} \int_a^b |g(x)|^q \, \mathrm{d}x = 1$$

So

$$\frac{1}{AB} \int_a^b |f(x)||g(x) \, \mathrm{d}x \leq 1$$

and rearranging yields the inequality.

□

Proof. Second method.
$a = x_0 < x_1 < \ldots < x_n = b$ partition of $[a, b]$. Define $\Delta x_k = x_{k+1} - x_k$ for $k = 0, \ldots, n-1$.
Let $\varepsilon_k \in [x_k, x_{k+1}]$.

$$\begin{aligned} S_n(|f||g|) &= \sum_{k=0}^{n-1} |f(\varepsilon_k)||g(\varepsilon_k)| \Delta x_k \\ &= \sum_{k=0}^{n-1} |f(\varepsilon_k)| \Delta x_k^{1/p} |g(\varepsilon_k)| \Delta x_k^{1/q} \end{aligned}$$

using the Holder inequality for sequences,

$$\begin{aligned} &\leq \left(\sum_{k=0}^{n-1} |f(\varepsilon_k)|^p \Delta(x_k) \right)^{1/p} \left(\sum_{k=0}^{n-1} |g(\varepsilon_k)|^q \Delta x_k \right)^{1/q} \\ &= \left(S_n(|f|^p) \right)^{1/p} \left(S_n(|g|^q) \right)^{1/q} \end{aligned}$$

Take $n \to \infty$, $\max |\Delta x_k| \to 0$, Riemann sums converge to $\int_a^b$. So

$$\int_a^b |f(x)|g(x) \, \mathrm{d}x \leq \|f\|_p \|g\|q$$

□

§9 Tuesday, September 17: Tutorial II

§9.1 $\boxed{\textbf{The Cantor Set}}$

– Posesses deep and remarkable properties

– Discovered in 1874 by Henry Smith

– Mainly used by Georg Cantor in 1883

§9.1.1 Construction of the Cantor Set

$E_0 = [0, 1]$. $3^0 = 1$.

Divide the interval in three equal pieces and throw away the middle one: $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ $3^1 = 3$.
$E_2 = \ldots$ $3^2 = 9$.
In this way, we create a decreasing sequences of sets

$$E_0 \supset E_1 \supset \ldots \supset E_n \supset \ldots$$

E_n is the disjoint union of 2^n intervals, each with length 3^{-n}.

§9.1.2 The Cantor Ternery Set, K

$$K = \bigcap_{n=0}^{\infty} E_n$$

Theorem 9.1

1. *K is closed*
2. interior $K = \varnothing$
3. *K has no isolated points: $\forall x \in K, \forall \varepsilon > 0, \exists y \in K, y \neq x, y \in B(x, \varepsilon)$*
4. *$x \in K \iff x = \sum_{i=1}^{\infty} \frac{a_k}{3^k}$ where (a_k) is a sequence with each $a_k \in \{0, 2\}$.*
5. *K is uncountable*

Proof.

1. E_n is closed for each n so $K \bigcap_{n=0}^{\infty} E_n$ is closed as well.
2. (By contradiction). Assume interior $K \neq \varnothing$. So $\exists x \in$ interior K.

$$\implies \exists \delta > 0 \text{ s.t } \underbrace{B(x, \delta)}_{=(x-\delta, x+\delta),\ (\text{length } 2\delta)} \subset \text{interior } K \subset K$$

Choose $n \in \mathbb{N}$ large enough such that $3^{-n} < 2\delta$. Then $B(x, \delta) \subset E_n$. But E_n is a disjoint union of intervals of length $3^{-n} < 2\delta$. This is a contradiction.

3. Key observation: endpoints of intervals in E_n are not removed in subsequent E_m (for $m > n$). All endpoints, $\{0, 1, \frac{1}{3}, \frac{2}{3}, \ldots\} \subset K$.

Let $x \in K$ and $\varepsilon > 0$ be given. We choose n large enough so that $3^{-n} < \varepsilon$. $x \in E_n \implies x \in [a, b]$ for $[a, b] \subset E_n$. So $b - a < \varepsilon$.

$$\implies |x - b|, |x - a| < \varepsilon$$

We have found $y \neq x$ so that $y \in K$, $y \in B(x, \varepsilon)$.

4. We want to show $x \in K \iff x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ for $a_k \in \{0, 2\}$.

Consider $\frac{1}{3} \in K$. $\frac{1}{3} = \sum_{k=1}^{\infty} \frac{a_k}{3}$. If $a_1 = 1$ and $a_n = 0$ for $n > 1$, it seems like we have a contradiction. But we can choose other coefficients:

$$\sum_{k=2}^{\infty} \frac{2}{3^n} = 2\left(\frac{1}{9}\right) \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \left(\frac{1}{3}\right)$$

i.e ternary expansions are not unique.

Let $S = \{\sum_{k=1}^{\infty} \frac{a_k}{3^k} : a_k \in \{0, 2\}\}$.

If $x \in S$, then $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$. Let $x_n = \sum_{k=1}^{n} \frac{a_k}{3^k}$ be the n^{th} partial sum.

$x_1 = 0$ or $\frac{2}{3}$, either of which is a left endpoint of E_1.

Assume that x_{n-1} is a left endpoint of E_{n-1}.

Assume:

$$[x_{n-1}, x_{n-1} + \frac{1}{3^{n-1}}] \subset E_{n-1}$$

Then we can take out the middle third to obtain a subset of E_n:

$$[x_{n-1}, x_{n-1} + \frac{1}{3^n}] \cup [x_{n-1} + \frac{2}{3^n}, x_{n-1} + \underbrace{\frac{1}{3^{n-1}}}_{=\frac{3}{3^n}}] \subset E_n$$

So $x_n = x_{n-1} + \frac{a_n}{3^n}$ is a left endpoint of an interval in E_n.

Since $x_n \in K$, and K is closed and so contains all its limit points, $x \in K$. Thus, $S \subset K$.

To show that $K \subset S$, first notice that the intervals removed to obtain K are all of the form

$$I_{k,m} = \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \quad m \in \mathbb{N}, \quad 0 \le k \le 3^{m-1} - 1$$

Exercise: By induction on n, prove that $[0, 1] \setminus E_n = \bigcup_{m=1}^{n} \bigcup_{k=0}^{3^{m-1}-1} I_{k,m}$

$$K \subset S \iff x \notin S \implies x \notin K$$

(contrapositive).

Suppose $x \notin S$. Suppose x contains a '1' in its n^{th} digit, i.e.

$$x = \sum_{k=1}^{n-1} \frac{a_k}{3^k} + \frac{1}{3^n} + \sum_{k=n+1}^{\infty} \frac{a_k}{3^k}$$

and assume $\sum_{k=1}^{\infty} \frac{a_k}{3^k} < \frac{1}{3^n}$. (The condition on the 'tail sum' prevents the non-uniqueness of the ternary expansion from being a problem)

We rewrite

$$\sum_{k=1}^{n-1} \frac{a_k}{3^k} = \sum_{k=1}^{n-1} \frac{3(3^{n-k-1}a_k}{3^n} = \frac{3\lambda}{3^n}$$

where $\lambda = \sum k = 1^{n-1} 3^{n-k-1} a_k \in \mathbb{N} \cup \{0\}$.

Hence,

$$\frac{3\lambda + 1}{3^n} < x < \frac{3\lambda + 2}{3^n}$$

So $x \in I_{\lambda,n} \implies x \notin K$.

5. K is uncountable. i.e. $|K| = |[0, 1]|$. We already know $K \subset [0, 1]$, so we know $|K| \le |[0, 1]|$. Hence, we need to find a surjection from $K \to [0, 1]$ to complete the proof.

   We define a function $F : K \to [0, 1]$, F takes $x \in K$ and change all of its digits of its ternary expansion of '2's to '1's and then outputs the number with corresponding binary expansion. Since

$$K = \{\sum_{k=1}^{\infty} \frac{a_k}{3^k} : a_k \in \{0, 2\}\}$$

$$F(\sum_{k=1}^{\infty} \frac{a_k}{3^k}) = \sum_{k=1}^{\infty} \frac{a_k}{2} \frac{1}{2^k}$$

Note: the lower bound of F on K is 0 (all $a_k = 0$) and the upper bound is when all the a_k are 1s, $\sum \frac{1}{2^k} = 1$.

Let $y \in [0,1]$. Then $y = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$ for $b_k \in \{0,1\}$. Then for $x = \sum_{k=1}^{\infty} \frac{2b_k}{3^k}$, so $F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = y$.

So we are done.

□

§10 Thursday, September 19, 2013

Last class : Holder inequality on $C([a,b])$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_a^b |f(x)||g(x)| \,\mathrm{d}x \le \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p} \left(\int_a^b |g(x)|^q \,\mathrm{d}x\right)^{1/q}$$

Two proofs:

1. Mimicking finite sequence proof
2. Using finite sequence Holder inequality and Riemann sums

One does the same for Minkowski inequality:

$$\left(\int_a^b (f(x)+g(x))^p \,\mathrm{d}x\right)^{1/p} \le \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p} + \left(\int_a^b |g(x)|^p \,\mathrm{d}x\right)^{1/p}$$

for $p \ge 1$.

Two proofs are on the homework.

A consequence of the Minkowski inequality is the triangle inequality for $\|f\|_p = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}$ for $p \ge 1$.

Hence, $(C([a,b]), \|\cdot\|_p)$ is a normed space for $p \ge 1$.

When $p = 2$, the norm is induced by an inner product, $\langle f, g\rangle = \int_a^b fg \,\mathrm{d}x$. For $p > 2$, the parallelogram law fails, so $\|\cdot\|_p$ is not induced by an inner product.

Exercises: find an example of two functions f, g for which the paralellogram law fails for $p > 2$.

Proposition 10.1

$$\lim_{p\to\infty} \|f\|_p = \max_{a\le x\le b} |f(x)|$$

Notation: $\|f\|_\infty = \max_{a\le k\le b} |f(x)|$

Proof. Let $M = \max_{x\in[a,b]} |f(x)|$ (we know this exists, because f is continuous).

$$\|f\|_p = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p} \le \left(\int_a^b M^p \,\mathrm{d}x\right)^{1/p} = M(b-a)^{1/p}$$

So

$$\limsup_{p\to\infty} \|f\|_p \le M \limsup_{p\to\infty} (b-a)^{1/p} = M \limsup_{p\to\infty} e^{\frac{1}{p}\log(b-a)} = M$$

So we have $\limsup_{p\to\infty} \|f\|_p \le M$.

For the lower bound, $\exists x_0 \in [a,b]$ such that $M = \max_{a\le x\le b} |f(x)| = |f(x_0)|$.

Let $\varepsilon > 0$ be such that $M > \varepsilon$. By continuity of the function f, there exists a closed interval Δ in $[a,b]$ that contains x_0 and such that $|f(x)| \ge M - \varepsilon$ for $x \in \Delta$.

ε_i, x_0, f is continuous at x_0, so there is a $\delta > 0$ such that with $\Delta = [x_0, x_0+\delta]$ (if $x_0 = a$) or $\Delta = [x_0-\delta, x_0]$ (if $x_0 = b$) or $\Delta = [x_0-\delta, x_0+\delta]$ (if $a < x_0 < b$) s.t. for $x \in \Delta$, $|f(x) - f(x_0)| < \varepsilon$.

$$\varepsilon > |f(x) - f(x_0)| \geq M - f(x) \implies |f(x)| + |f(x_0) - f(x)| \geq |f(x) + f(x_0) - f(x)| = |f(x_0)|$$

$$|f(x)| \geq M - \varepsilon_1 \text{ for } x \in \Delta$$

So

$$\begin{aligned}\left(\int_a^b |f(x)|^p \, \mathrm{d}x\right)^{1/p} &\geq \left(\int_\Delta |f(x)|^p \, \mathrm{d}x\right)^{1/p} \\ &\geq \left(\int_\Delta (M-\varepsilon)^p \, \mathrm{d}x\right)^{1/p} \\ &= (M-\varepsilon)|\Delta|^{1/p}\end{aligned}$$

where $|\Delta|$ is the length of the interval.

Then

$$\liminf_{p\to\infty} \|f\|_p \geq (M-\varepsilon) \liminf_{p\to\infty} |\Delta|^{1/p} \geq M - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $\liminf_{p\to\infty} \|f\|_p \geq M$.

So we have an upper and lower bound of M, so M is the limit. □

$C([a,b])$, $\|\cdot\|_p$, $p \in [1,\infty]$. $\|f\|_p$, $\|f\|_\infty$.

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, \mathrm{d}x\right)^{1/p} \leq \|f\|_\infty (b-a)^{1/p}$$

Are the norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ equivalent? In other words, can we find a constant D such that for all $f \in C([a,b])$,

$$\|f\|_p \geq D\|f\|_\infty$$

Let's define a sequence of functions $f_n(x)$ on the interval $[a,b]$. $f_n(a) = n$, then it linearly descends to 0 at a point ($f_n(a_n) = 0$) for $a_n = a + \frac{b-a}{n^{2p}}$.

$$\int_a^b |f_n(x)|^p \, \mathrm{d}x \leq \int_a^{a_n} n^p \, \mathrm{d}x = n^p(a_n - a) = \frac{n^p(b-a)}{n^{2p}} \to 0$$

$$\|f_n\|_\infty \to \infty \text{ as } n \to \infty$$

(Exercise: Show that $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent if $p \neq q$).

So infinite dimensional analysis is much richer that finite dimensional.

§10.1 Ch. 9.3: Continuous mappings between metric spaces

Two metric spaces (X, d_x), (Y, d_y), and a function $f : X \to Y$.

Definition 10.2 The function f is continuous at a point $x \in X$ if for any sequence $(x_n) \in X$ such that $(x_n) \to x$ (i.e. $\lim_{n\to\infty} d_X(x_n, x) = 0$), we have $(f(x_n)) \to f(x)$ in Y (that is, $\lim_{n\to\infty} d_Y(f(x_n), f(x)) = 0$).

A function is continuous on a set $A \subset X$ if it is continuous at every point in A.

f is called continuous if it is continuous at every point $x \in X$.

Proposition 10.3 TFAE:

1. f is continuous at $x \in X$
2. $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall y \in X)(d_X(x,y) < \delta) \implies d_Y(f(x), f(y)) < \varepsilon)$

Remark. In other words, f is continuous at $x \iff$ for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_x(x,\delta)) \subset B_Y(f(x), \varepsilon)$.

Proof. (1 $\implies$ 2) Suppose (1) holds and (2) fails. (2) fails means

$$(\exists\varepsilon > 0)(\forall\delta > 0)(\exists y \in X)(d_x(x, y) < \delta \text{ and } d_Y(f(x), f(y)) \geq \varepsilon$$

Fix such an $\varepsilon_0 > 0$. Take $\delta_n = \frac{1}{n}$ and let $y_n \in B_x(x, \frac{1}{n})$ be such that $d_Y(f(x), f(y_n)) \geq \varepsilon_0$. Then $d(x, y_n) < \frac{1}{n}$ and $d_Y(f(x), f(y_n)) \geq \varepsilon_0$.
So

$$\lim_{n\to\infty} d_X(x, y_n) = 0 \text{ and } \liminf_{n\to\infty} d_Y(f(x), f(y_n)) \geq \varepsilon_0 > 0$$

This contradicts (1).
Now we will show (2) $\implies$ (1).
We need to show that for any sequence (x_n) satisfying $\lim_{n\to\infty} d_X(x_n, x) = 0$, we have

$$\lim_{n\to\infty} d_Y(f(x_n), f(x)) = 0$$

Let (x_n) be such a sequence. Let $\varepsilon > 0$. Then by (2), $\exists\delta > 0$ such that

$$d_x(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Since $\lim_{n\to\infty} d_X(x_n, x) = 0$, we have $\exists N$ such that for all $n \geq N$, $d_X(x, x_n) < \delta$. But then for $n \geq N$, $d_Y(f(x), f(x_n)) < \varepsilon$.
This means that $\lim_{n\to\infty} d_Y(f(x), f(x_n)) = 0$. So f is continuous at x.

□

Proposition 10.4 TFAE:

1. f is a continuous function
2. For any open set $O \subset Y$, $f^{-1}(O) = \{x : f(x) \in O\}$ is an open set in X.

§11 Friday, September 20, 2013

End of last class: $(X, d_X), (Y, d_Y)$ metric spaces, $f : X \to Y$.

Proposition 11.1 TFAE:

1. f is a continuous function
2. For any open set $O \subset Y$, $f^{-1}(O) = \{x : f(x) \in O\}$ is an open set in X.

Proof. (1) $\implies$ (2): Let O be open in Y. If $f^{-1}(O)$ is empty set, then $f^{-1}(O)$ is open. Otherwise, let $x \in f^{-1}(O)$. Then $f(x) \in O$, and since O is open, $\exists\varepsilon > 0$ such that $B_Y(f(x), \varepsilon) \subset O$.

Since f is continuous, we can find $\delta > 0$ such that $f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon) \subset O$. Hence, $B_X(x, \delta) \subset f^{-1}(O)$. This implies that $f^{-1}(O)$ is open.

(2) $\implies$ (1): We need to show that for all $x \in X$,

$$(\forall\varepsilon > 0)(\exists\delta > 0)(f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon))$$

Let x be given and $\varepsilon > 0$ be given.
Then $B_Y(f(x), \varepsilon)$ is an open set in Y. So by (2),

$$f^{-1}(B_Y(f(x), \varepsilon))$$

is an open set in X. Obviously, $x \in f^{-1}(B_Y(f(x), \varepsilon))$.
So $\exists\delta > 0$ such that

$$B_X(x, \delta) \subset f^{-1}(B_Y(f(x), \varepsilon))$$

and hence we have

$$f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon)$$

and so f is a continuous function.

□

Proposition 11.2 $(X, d_X), (Y, d_Y), (Z, d_Z)$ metric spaces, and let $f : X \to Y$, $g : Y \to Z$ be continuous functions.

Then

$$h = g \circ f : X \to Z$$

is continuous.

Proof. Let $O \subset Z$ be open. Then $h^{-1}(O) = f^{-1}(g^{-1}(O))$. f, g are continuous, so $g^{-1}(O)$ is open in Y, and so $h^{-1}(O) = f^{-1}(g^{-1}(O))$ is open in X. Hence, h is continuous. □

§11.1 Uniform continuity

Continuity says for $f : X \to Y$

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in X)(d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$$

Uniform continuity just switches the order of a quantifier:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$$

Now, δ does not depend on x.

Example 11.3 (Analysis 2): $f(x) = {}^1\!/\!_x$ in $(0, 1]$ is not uniformly continuous.

Example 11.4 Let X be a nonempty set and d the discrete metric on X.

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Let (Y, d_Y) be any other metric space. Then any function $f : X \to Y$ is uniformly continuous. We need to show:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$$

So take $\delta = \frac{1}{2}$. Then $d_x(x, y) < \frac{1}{2} \implies x = y \implies d_Y(f(x), f(y)) = 0 < \varepsilon$.

Definition 11.5 A function $f : X \to Y$ is called Lipschitz if $\exists C > 0$ such that $\forall x, y \in X$,

$$d_Y(f(x), f(y)) \leq C d_X(x, y)$$

This is a stronger notion than uniform continuity: if f is Lipschitz, then f is uniformly continuous.

$$(\forall \varepsilon > 0)(\delta = \frac{\varepsilon}{C})(\forall x, y \in X)(d_X(x, y) < \delta = \frac{\varepsilon}{C} \implies d_Y(f(x), f(y)) \leq C d_X(x, y) < \varepsilon$$

From analysis 2: $f : [a, b] \to \mathbb{R}$, f differentiable, $|f'(x)|$ bounded,

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|$$

where $M = \max_{x \in [a,b]} |f'(x)|$.

Example 11.6 *(1)* $(C([a, b], \|\cdot\|_p), p \in [1, \infty]$.

$$F : C([a, b]) \to \mathbb{R}$$

$$F(f) = \int_a^b f(x)\,\mathrm{d}x$$

If $1 \leq p < \infty$,

$$\begin{aligned} |F(f) - F(g)| = \left| \int_a^b (f - g)\,\mathrm{d}x \right| &\leq \int_a^b |f - g|\,\mathrm{d}x \\ &\leq \left(\int_a^b |f - g|^p\,\mathrm{d}x \right)^{1/p} \left(\int_a^b 1\,\mathrm{d}x \right)^{1/q} \\ &= \|f - g\|_p (b - a)^{1/q} \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

So, F is Lipschitz, hence uniformly continuous. If $p = \infty$, we have the same result,

$$|F(f) - F(g)| \leq \|f - g\|_\infty (b - a)$$

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$

F is a linear operator

Example 11.7 *(2)* $(C([a,b], \|\cdot\|_p)$, $p \in [1, \infty]$.

$$F : C([a,b]) \to \mathbb{R}$$

$$F(f) = \int_a^b (f(x))^2 \,\mathrm{d}x$$

F is *not* linear. For what values of p is this function continuous?

Case 1: $2 < p < \infty$. ($p = 2$ and $p = \infty$ is an exercise)

$$\begin{aligned}\left|F(f) - F(g)\right| &= \left|\int_a^b (f(x)^2 - g(x)^2)\,\mathrm{d}x\right| = \left|\int_a^b (f(x) - g(x))(f(x) + g(x))\,\mathrm{d}x\right| \\ &\leq \left(\int_a^b |f - g|^p \,\mathrm{d}x\right)^{1/p} \underbrace{\left(\int_a^b |f(x) + g(x)|^q \,\mathrm{d}x\right)^{1/q}}_{\text{need a control on this}}\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. We must be careful in infinite dimensional spaces.

We will use $2 \leq p < \infty$. Let $qq' = p$. We will do Holder inequality with q', p', $\frac{1}{p'} + \frac{1}{q'} = 1$.

$qq' = p$. $\frac{1}{p} + \frac{1}{q} = 1$. So $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$. So $q = \frac{p}{p-1}$. $qq' = p \implies \frac{p}{p-1} q' = p$. So $q' = p - 1$. $p > 2$ so $q' > 1$.

$$\left(\int_a^b |f(x) + g(x)|^q difx\right)^{1/q} \leq \left(\int_a^b |f(x) + g(x)|^{qq'} \,\mathrm{d}x\right)^{1/_{q'q}} \left(\int_a^b 1 \,\mathrm{d}x\right)^{1/_{p'q}}$$

So continuing from before,

$$\left|F(f) - F(g)\right| \leq \|f - g\|_p \left(\int_a^b |f(x) + g(x)|^p\right)^{1/_{qq'}} (b - a)^{1/_{p'q}} = \|f - g\|_p (\|f + g\|_p)^{p/_{qq'}} (b - a)^{1/_{p'q}}$$

Let f_n be such that $f_n \to g$ in $(C([a,b]), \|\cdot\|_p)$.

$$|F(f_n) - F(g)| \leq \|f_n - g\|_p (\|f_n\|_p + \|g\|_p)(b - a)^{1/_{p'q}}$$

f_n is a convergence sequence, so $\|f_n\|$ is bounded, $\|f_n\|_p \leq M$ for all n.

$$|F(f_n) - F(g)| \leq \|f_n - g\|_p C$$

for some constant C.

Hence, $\lim_{n\to\infty} \|f_n - g\|_p C \to 0$, so F is continuous. But not uniformly continuous! C depends on g.

For $1 \leq p < 2$, $F(f) = \int_a^b f(x)^2 \,\mathrm{d}x$ is *not* continuous on $(C([a,b]), \|\cdot\|_p)$.

Example 11.8 (in response to a student's question)

Something that is Lipschitz but not continuous: $\mathbb{R}$ with discrete metric, $f(x) = x$, $f : \mathbb{R} \to \mathbb{R}$. Any function is uniformly continuous with discrete metric, but it's not Lipschitz: If $|f(x) - f(y)| \leq Cd(x,y) \leq C$ then f would have to be bounded. But $f(x)$ is unbounded on $\mathbb{R}$.

§12 Tuesday, September 24, 2013

- Check course website!
- Extension to homework 2: due date October 1, in class
- New time for tutorial: Friday 13:30-15:30, Wong 150
- Yaril's new office hours: Thursday's, 2pm-6pm Burn 1018

Last class (redoing it because Prof. Jaksik got mixed up)

Example 12.1 $(C([a,b]), \|\cdot\|_p)$, $1 \le p \le \infty$, called L^p−space. $F : C([a,b]) \to \mathbb{R}$, $F(f) = \int_a^b [f(x)]^2 \,\mathrm{d}x$. For what values of p is the map F continuous?

$$\begin{aligned} |F(f) - F(g)| &= \left| \int_a^b [f(x)]^2 - [g(x)]^2 \,\mathrm{d}x \right| = \left| \int_a^b (f(x) - g(x))(f(x) + g(x)) \,\mathrm{d}x \right| \\ &\le \int_a^b |f(x) - g(x)||f(x) + g(x)| \,\mathrm{d}x \qquad (\star) \end{aligned}$$

First we will deal with $2 \le p < \infty$ ($p = \infty$ as exercise). If $p = 2$, we can just apply Cauchy-Schwarz:

$$\begin{aligned} \int_a^b |f(x) - g(x)||f(x) + g(x)| \,\mathrm{d}x &\le \left(\int_a^b |f(x) - g(x)|^2 \,\mathrm{d}x \right) \left(\int_a^b |f(x) + g(x)|^2 \,\mathrm{d}x \right) \\ &= \|f - g\|_2 \|f + g\|_2 \le \|f - g\|_2 (\|f\|_2 + \|g\|_2) \end{aligned}$$

If $2 < p < \infty$,
Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Apply to $(\star)$ Holder inequality:

$$\int_a^b |f(x) - g(x)||f(x) + g(x)| \,\mathrm{d}x \le \left(\int_a^b |f(x) - g(x)|^p \,\mathrm{d}x \right)^{1/p} \left(\int_a^b |f(x) + g(x)|^q \,\mathrm{d}x \right)^{1/q}$$

Let q' be such that $q'q = p$ and p' by $\frac{1}{q'} + \frac{1}{p'} = 1$. Then $q' = p - 1$.
So then

$$\int_a^b |f(x) + g(x)|^q \,\mathrm{d}x \le \left(\int_a^b |f(x) + g(x)|^{qq'} \,\mathrm{d}x \right)^{1/q'} \left(\int_a^b 1 \,\mathrm{d}x \right)^{1/p'}$$

So we have

$$\left(\int_a^b |f(x) + g(x)|^q \,\mathrm{d}x \right)^{1/q} \le \left(\int_a^b |f(x) + g(x)|^{qq'} \,\mathrm{d}x \right)^{1/qq'} (b - a)^{1/p'q}$$

So, for $2 < p < \infty$,

$$|F(f) - F(g)| \le \int_a^b |f(x) - g(x)||f(x) + g(x)| \,\mathrm{d}x \le \|f - g\|_p \|f + g\|_p (b - a)^{1/p'q}$$

So, for $2 \le p < \infty$,

$$|F(f) - F(g)| \le C_p \|f - g\|_p (\|f\|_p + \|g\|_p)$$

where $C_p = \begin{cases} 1 & \text{if } p = 2 \\ (b-a)^{1/p'q} & \text{if } p > 2 \end{cases}$.

We will now show that F is continuous for $2 \le p < \infty$. To prove that, one needs to show that if $(f_n)_{n=1}^{\infty}$ is a sequence in $C([a,b])$ that converges to g in $\|\cdot\|_p$ norm, then $\lim_{n\to\infty} |F(f_n) - F(g)| = 0$.

We have the estimate,

$$|F(f_n) - F(g)| \le C_p \|f_n - g\|_p (\|f_n\|_p + \|g\|_p)$$

We know (f_n) is a convergent sequence and so is bounded and g is fixed. So $\sup_n \|f_n\| = M < \infty$. Then we have

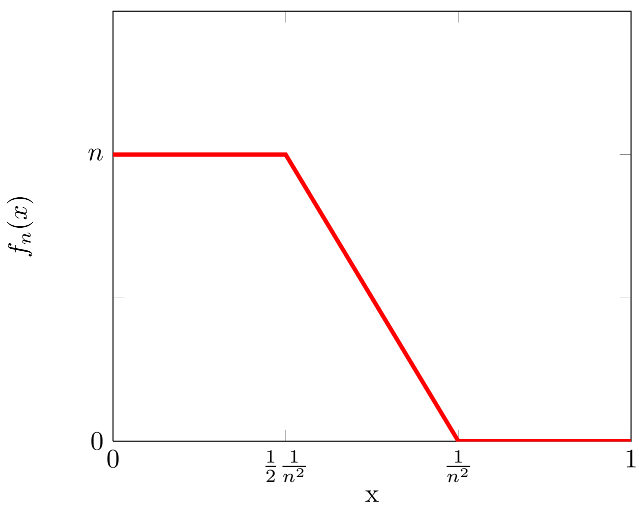
$$|F(f_n) - F(g)| \le C_p \|f - n - g\|_p (M + \|g\|_p)$$

So, $\lim_{n\to\infty} F(f_n) = F(g)$ whenever $(f_n) \to g$ in $(C([a,b]), \|\cdot\|_p)$. So F is continuous if $2 \le p < \infty$.

If $1 \le p < 2$, then F is not continuous.

For notational simplicity, take $a = 0$, $b = 1$.

Then consider the following function:



$$\int_0^1 |f_n(x)|^p \, \mathrm{d}x \le \int_0^{1/n^2} n^p \, \mathrm{d}x = \frac{n^p}{n^2} \to 0 \text{ as } n \to \infty$$

(if $1 \le p < 2$).

So $\|f_n\|_p \to 0$ as $n \to \infty$.

$$F(f_n) = \int_0^1 [f_n(x)]^2 \, \mathrm{d}x \ge \int_0^{\frac{1}{2}\frac{1}{n^2}} (f_n(x))^2 \, \mathrm{d}x = \int_0^{\frac{1}{2}\frac{1}{n^2}} n^2 \, \mathrm{d}x = \frac{1}{2}$$

So we have that $f_n \to 0$ in $\|\cdot\|_p$, $1 \le p < 2$, but

$$\liminf_{n\to\infty} F(f_n) \ge \frac{1}{2} + F(0)$$

So F is not continuous at 0 and hence is not continuous.

§12.1 9.4: Completeness

Definition 12.2 Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is called Cauchy if

$$(\forall \varepsilon > 0)(\exists N > 0)(\forall n, m \ge N)(d(x_n, x_m) < \varepsilon)$$

Proposition 12.3 If (x_n) is convergent, then (x_n) is Cauchy.

Proof. Suppose $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Then, $\exists N > 0$ such that for all $n \geq N$,

$$d(x_n, x) \leq \frac{\varepsilon}{2}$$

Then for $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Proposition 12.4 If a Cauchy sequence has a convergent subsequence, then the Cauchy sequence is convergent.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence, and let $(x_{n_k})_{k=0}^{\infty}$ be a convergent subsequence,

$$\lim_{k\to\infty} x_{n_k} = x$$

$(n_1 < n_2 < \ldots < n_k < \ldots)$

Given that (x_n) is Cauchy, given $\varepsilon > 0$, we can find $N > 0$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$, for $n, m \geq N$. Since

$$\lim_{n\to\infty} d(x_{n_k}, x) = 0$$

we can find some index $n_{k_0} > N$ such that $d(x_{n_{k_0}}, x) < \frac{\varepsilon}{2}$.

Then, for $n \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

since $n_{k_0} > N$ and $d(x_{n_{k_0}}, x) < \frac{\varepsilon}{2}$.

□

Remark. It is *not* true that every Cauchy sequence is convergent. Metric spaces in which every Cauchy sequence is convergent are called *complete*, and play a very important role in analysis.

Definition 12.5 A metric space (X, d) is called complete if every Cauchy sequence in X converges to an element of X.

Example 12.6 $\mathbb{R}$, $d(x, y) = |x - y|$ is complete. Very non-trivial to show: we have the Peano axioms for $\mathbb{N}$, which we can extend to $\mathbb{Z}$ and then $\mathbb{Q}$. Then we have to "complete" $\mathbb{Q}$. We define $\mathbb{R}$ as the collection of sequences under some equivalence relation. (the tutorial will go in greater detail).

§13 Thursday, September 26, 2013

§13.1 Banach spaces

Recall: A sequence $(x_n)_{n=1}^{\infty}$ in X is called Cauchy if

$$(\forall \varepsilon > 0)(\exists N > 0)(\forall n, m \geq N)(d(x_n, x_m) < \varepsilon)$$

Example 13.1 Let X be a non-empty set, and d the discrete metric:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Let (x_n) be a Cauchy sequence in (X, d). Take $\varepsilon = 1/2$. Then there is a $N > 0$ such that $\forall m, n \geq N$, $d(x_m, x_n) =< 1/2$. So $x_m = x_n$ for $m, n \geq N$.

That is, $x_N = x_{N+1} = x_{N+2} = \ldots$.

So a Cauchy sequence in (X, d) is eventually constant, i.e. all the elements are equal starting with a large enough index. In particular, every Cauchy sequence is convergent. So (X, d) is complete.

Other things about the discrete metric:

– Any point is an open set
  $B(x, 1/2) = \{y \in X : d(x,y) < 1/2\} = \{x\}$

– If $A \subset X$ then $A = \bigcup_{x \in A} \{x\}$, so A is open.

– But then A^c is also open! So A is closed.

– So in (X, d), any set is both open and closed. Such a space is called totally disconnected in topology.

Definition 13.2 Let now $(X, \|\cdot\|)$ be a normed space. If the induced metric

$$d(x,y) = \|x - y\|$$

is complete, then $(X, \|\cdot\|)$ is called a Banach space.

Definition 13.3 If $(X, \|\cdot\|)$ is a Banach space and $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot \rangle$,

$$\|x\| = \sqrt{\langle x, x \rangle}$$

for $x \in X$. Then X is called a Hilbert space.

"Quantum mechanics = Hilbert spaces"

Example 13.4 $(C([a,b]), \|\cdot\|_\infty)$ is a Banach space.

Proof. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(C([a,b]), \|\cdot\|_\infty)$.
Then $(\forall \varepsilon > 0)(\exists N > 0)(\forall n, m \geq N)(\|x_n - x_m\|_\infty = \max_{a \leq x \leq b} |f_m(x) - f_n(x)| < \varepsilon)$.
We want to prove that (f_n) is convergent, that is, $\exists\, f \in C([a,b])$ such that

$$\lim_{n \to \infty} \|f_n - f\|_\infty = 0$$

It follows that for any $x \in [a,b]$,

$$(\forall \varepsilon > 0)(\exists N > 0)(\forall m, n \geq N)(|f_m(x) - f_n(x)| < \varepsilon)$$

This means that for any $(f_m(x))$ is a Cauchy sequence of real numbers, and we know that the real line is complete:

$$\lim_{n \to \infty} f_n(x) = f(x) \qquad \forall x \in [a,b]$$

We need to prove two things:

1. f is a continuous function, so $f \in C([a,b])$
2. $f_n \to f$ in $(C([a,b]), \|\cdot\|_\infty)$

To prove (1), we want to show that for any $x_0 \in [a,b]$, and any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y \in [a,b]$ satisfying $|x_0 - y| < \delta$, we have $|f(x_0) - f(y)| < \varepsilon$.
Let $x_0 \in [a,b]$ be given, and let $\varepsilon > 0$.
We know that there is an $N \geq 0$ such that $\forall m, n \geq N$,

$$\max_{x \in [a,b]} |f_m(x) - f_n(x)| < \frac{\varepsilon}{3}$$

So for any $x \in [a,b]$ and all $m, n \geq N$,

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{3}$$

So for fixed $m \geq N$, we can take $n \to \infty$ and get that

$$|f_m(x) - f(x)| \leq \frac{\varepsilon}{3}$$

for all $m \geq N$.

In particular, for $m = N$ and all $x \in [a, b]$,

$$|f_N(x) - f(x)| \leq \frac{\varepsilon}{3}$$

$$\begin{aligned} |f(x_0) - f(y)| &= |f(x_0) - f_N(x_0) + f_N(x_0) - f_N(y) + f_N(y) - f(y)| \\ &\leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(y)| + |f_N(y) - f(y)| \\ &\leq \frac{2\varepsilon}{3} + |f_N(x_0) - f_N(y)| \end{aligned}$$

Since f_N is continuious, $\exists \delta > 0$ such that if $y \in [a, b]$ and $|x_0 - y| < \delta$, then

$$|f_N(x_0) - f_N(y)| < \frac{\varepsilon}{3}$$

So for such y

$$|f(x_0) - f(y)| < \varepsilon$$

So f is continuous.

Now, to prove (2):

Given $\varepsilon > 0$, $\exists N > 0$, $\forall m, n \geq N$,

$$\|f_m - f_n\|_\infty = \max_{x \in [a,b]} |f_m(x) - f_n(x)| < \frac{\varepsilon}{2}$$

$\forall x \in [a, b]$,

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2}$$

Take $m \to \infty$. Then $\forall x \in [a, b]$,

$$|f(x) - f_n(x)| \leq \frac{\varepsilon}{2}$$

for $n \geq N$.

So, for $n \geq N$,

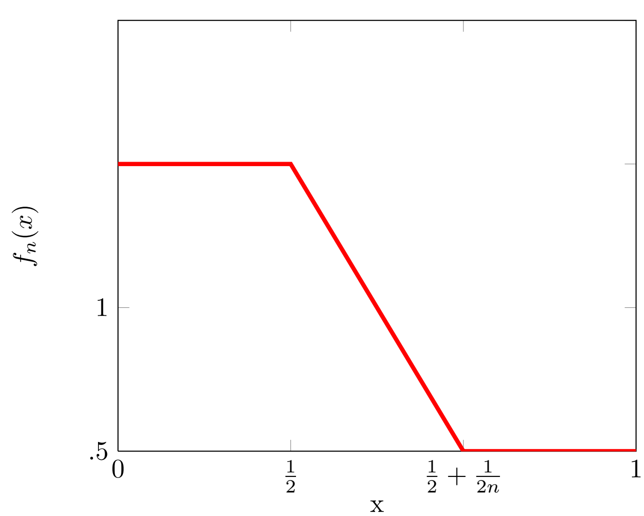
$$\|f_n - f\|_\infty = \max_{x \in [a,b]} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

□

Example 13.5 $(C([a, b]), \| \cdot \|_p)$ for $1 \leq p < \infty$ is *not* complete.

Proof. Take $a = 0, b = 1, p = 1$ (the rest is an exercise).

Define a sequence of functions (f_n) by the following plot:



Consider $f_n(x) - f_m(x)$. By looking at the graph, we can see

$$-1 \leq f_n(x) - f_m(x) \leq 1$$

for $x \in [0, 1]$.

Also, $f_m(x) - f_n(x) = 0$ if $0 \leq x \leq \frac{1}{2}$.

Additionally, $f_m(x) - f_n(x) = 0$ if $x \geq \frac{1}{2} + \max\left(\frac{1}{2n}, \frac{1}{2m}\right)$.

So,

$$\int_0^1 |f_n(x) - f_m(x)| \, \mathrm{d}x = \int_{1/2}^{\frac{1}{2}+\max\left(\frac{1}{2n},\frac{1}{2m}\right)} |f_n(x) - f_m(x)| \, \mathrm{d}x \leq \max\left(\frac{1}{2n}, \frac{1}{2m}\right)$$

So if $m, n \geq N$, then

$$\int_0^1 |f_m(x) - f_n(x)| \, \mathrm{d}x \leq \frac{1}{2N}$$

So this sequence (f_n) is Cauchy in $(C([a, b]), \| \cdot \|)$.

Suppose now that (f_n) converges to g in $(C([a, b]), \| \cdot \|)$.

In particular, g is continuous.

$$\lim_{n\to\infty} \int_0^1 |f_n - g| \, \mathrm{d}x = 0$$

Then,

$$\lim_{n\to\infty} \int_0^{1/2} |f_n(x) - g(x)| \, \mathrm{d}x = 0 = \int_0^{1/2} |1 - g| \, \mathrm{d}x$$

so, $g(x) = 1$ for $0 \leq x \leq 1/2$.

For some t such that $1 > t > 1/2$.

$$\int_0^1 |f_n - g| \, \mathrm{d}x \geq \int_t^1 |f_n - g| \, \mathrm{d}x$$

So,

$$\lim_{n\to\infty} \int_t^1 |f_n - g| \, \mathrm{d}x = 0 = \int_t^1 |g| \, \mathrm{d}x = 0$$

For n large enough, $f_n(x) = 0$ for $x \in [t, 1]$.

So $g(x) = 0$ for $x \in [t, 1]$ and any $t > 1/2$. So $g(x) = 0$ for $x > 1/2$.

$$g(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{if } 1/2 < x \leq 1 \end{cases}$$

So g is *not* continuous, so f_n cannot converge. □

§14 Friday, September 27, 2013

Important correction: Tutorials are in room Wong 1050, not 150.

Proposition 14.1 Let (Y, d) be a metric subspace of a complete metric space (X, d).
Then (Y, d) is complete if and only if Y is a closed subset of X.

Proof. Suppose first that Y is closed.

Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in Y. Then (x_n) is also a Cauchy sequence in X. Since X is complete, $(x_n) \to x \in X$. But since Y is closed, $x_n \in Y$, then $x \in \operatorname{cl} Y = Y$.

So every Cauchy sequence in Y converges to an element of Y, so Y is complete.

Conversely, assume that (Y, d) is complete.

Let $x \in \operatorname{cl} Y$. Then $\exists (x_n)$ with $x_n \in Y$ such that $(x_n) \to x$. Since (x_n) is convergent, (x_n) is also Cauchy. Since (Y, d) is complete (x_n) has to converge to an element of Y; by the uniqueness of limits then, $x \in Y$. Hence, $\operatorname{cl} Y \subset Y$, and Y is closed. □

Definition 14.2 Let Y be a non-empty subset of metric space (X, d). The diameter of Y, denoted $\operatorname{diam}(Y)$ is defined by
$$\operatorname{diam}(Y) = \sup\{d(x, y) : x, y \in Y\}$$

Definition 14.3 If X is a metric space then $Y \subset X$ is called bounded if $\operatorname{diam}(Y) < \infty$.

Proposition 14.4
$$\operatorname{diam}(Y) = \operatorname{diam}(\operatorname{cl}(Y))$$

Proof. Since $\operatorname{cl} Y \supset Y$, it is obvious that
$$\operatorname{diam}(\operatorname{cl}(Y)) \geq \operatorname{diam} Y$$

Let now $x, y \in \operatorname{cl} Y$. Then we can find two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ such that $(x_n) \to x$ and $(y_n) \to y$, that is,
$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(y_n, y) = 0$$

We have that $d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$, so
$$d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y, y_n)$$

But also
$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$
and so
$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y, y_n)$$

So,
$$\big|d(x, y) - d(x_n, y_n)\big| \leq d(x, x_n) + d(y, y_n)$$

thus,
$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$$

$$\operatorname{diam}(Y) = \sup\{d(x, y) : x, y \in Y\} \geq d(x_n, y_n)$$
for all n.
$$\operatorname{diam}(Y) \geq \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$$

So,
$$\operatorname{diam}(Y) \geq d(x, y)$$
for all $x, y \in \operatorname{cl} Y$.

So, $\operatorname{diam}(Y)$ is an upper bound for the set
$$\{d(x, y) : x, y \in \operatorname{cl} Y\}$$

hence, since sup is the least upper bound,
$$\operatorname{diam}(Y) \geq \sup\{d(x, y) : x, y \in \operatorname{cl} Y\} = \operatorname{diam}(\operatorname{cl} Y)$$

□

Definition 14.5 A descending sequence Y_n, $n = 1, 2, \ldots$
$$(Y_1 \supset Y_2 \supset \ldots \supset Y_n \supset \ldots)$$
of non-empty subsets of a metric space (X, d) is called contracting if
$$\lim_{n\to\infty} \operatorname{diam}(Y_n) = 0$$

Theorem 14.6 *(Cantor Intersection Theorem) T.F.A.E.:*

1. *Metric space (X, d) is complete*

2. For any contracting sequence F_n of non-empty closed subsets of X, there exists $x \in X$ such that

$$\bigcap_{n=1}^{\infty} F_n = \{x\}$$

Proof. (1) $\implies$ (2): Let F_n be as in (2).

Choose $x_n \in F_n$. We claim that the sequence (x_n) is Cauchy. To prove that, let $\varepsilon > 0$.

Since $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, there exists N such that

$$\operatorname{diam}(F_N) < \varepsilon$$

Since for $m, n \geq N$, $x_n \in F_n \subset F_N$, $x_m \in F_m \subset F_N$,

$$d(x_m, x_n) \leq \operatorname{diam}(F_N) < \varepsilon$$

So the sequence (x_n) is Cauchy, and

$$\lim_{n\to\infty} x_n = x$$

by assumption (1). Now, $x_n \in F_n \subset F_1$ for all n, so $(x_n)_{n=1}^{\infty}$ is a sequence in the closed subset F_1, so $x \in F_1$.

But $(x_n)_{n=2}^{\infty}$ is a sequence in F_2, which is closed, so $x \in F_2$.

For any k, $(x_n)_{n=k}^{\infty}$ is a sequence in F_k which is closed, and so $x \in F_k$.

So $x \in F_k$ $\forall k$, and hence, $\{x\} \subset \bigcap_{k=1}^{\infty} F_k$.

If $y \in \bigcap_{k=1}^{\infty} F_k$, then $x, y \in F_k$ for all k, and $d(x, y) \leq \operatorname{diam}(F_k)$.

Since F_k is contracting,

$$\lim_{k\to\infty} \operatorname{diam}(F_k) = 0$$

and $d(x, y) = 0$, that is, $x = y$.

Thus,

$$\bigcap_{k=1}^{\infty} F_k = \{x\}$$

Now, (2) $\implies$ (1) : Let (x_n) be a Cauchy sequence in X.

---

aside General idea: define F_k in such a way that we can exploit (2) to show (x_n) converges.

Let

$$F_k = \operatorname{cl}\{x_n : n \geq k\}$$

F_k is obviously closed. Since

$$\{x_n : n \geq k\} \supset \{x_n : n \geq k+1\}$$

So,

$$F_{k+1} \supset F_k$$

Let $\varepsilon > 0$. Since (x_n) is Cauchy, $\exists N > 0$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon$.

$$\operatorname{diam}\{x_n : n \geq N\} = \sup\{d(x_n, x_m) : n, m \geq N\} \leq \varepsilon$$

Then $\operatorname{diam} \operatorname{cl}\{x_n : n \geq N\} = \operatorname{diam}\{x_n : n \geq N\}$, so $\operatorname{diam} F_N \leq \varepsilon$. Since F_n is descending, $F_n \subset F_N$ is $n \geq N$, we have

$$\operatorname{diam}(F_n) \leq \varepsilon$$

for $n \geq N$. So,

$$\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$$

So F_n is contracting.

So, by (2),

$$\bigcap_{k=1}^{\infty} F_k = \{x\}$$

For any k, $x_k \in F_k$ and $x \in F_k$.

$$d(x_k, x) \leq \operatorname{diam} F_k$$

and $\lim_{k\to\infty} \operatorname{diam} F_k = 0$.

So,

$$\lim_{k\to\infty} d(x_k, x) = 0$$

So our sequence converges. Hence, X is complete.

□

§14.1 Completion of metric spaces

§14.1.1 Few definitions:

Definition 14.7 Two metric spaces (X, d_X), (Y, d_Y) are called isometric if there exists a bijection $f : X \to Y$ such that $\forall x, y \in X$,

$$d_Y(f(x), f(y)) = d_X(x, y)$$

aside This is an equivalence relationship on the collection of all metric spaces

One identifies isometric metric spaces.

Definition 14.8 A subset S of a metric space (X, d) is called dense if

$$\operatorname{cl}(S) = X$$

In other words, S is dense in X if for any $x \in X$, there is a sequence (x_n) in S such that $\lim_{n\to\infty} x_n = x$.

Example 14.9 Rationals are dense in $\mathbb{R}$.

Exercise: Let d be a discrete metric on a set X. Show that if S is dense in X then $S = X$.

Definition 14.10 A complete metric space (Y, d_Y) is called a completion of a metric space (X, d_X) if there exists a subset $\tilde{Y} \subset Y$ such that

1. $\tilde{Y}$ is dense in Y
2. There exists a bijection

$$f : X \to \tilde{Y} \text{ such that } \forall x, y \in X, d_Y(f(x), f(y)) = d_X(x, y)$$

In other words, (Y, d_Y) is a completion of (X, d_X) if (X, d_X) is isometric to a dense subspace of Y.

Theorem 14.11

1. *Any metric space has a completion*
2. *A completion of a metric space is unique up to an isometry*

   In other words, if (Y, d_Y) and (Z, d_Z) are completions of (X, d_X), then they are isometric spaces.
3. *Completion of a normed space is a Banach space*
4. *If in (3) the norm of X was induced by an inner product, then completed space is also induced by an inner product. (i.e. the completion of an inner product space is an Hilbert space)*

§15 Friday, September 27, 2013: Tutorial III

§15.1 Completion of metric spaces

Theorem 15.1 *Any metric space has a completion*

First, a lemma:

Lemma 15.2 *Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two Cauchy sequences. Then*

$$\lim_{n\to\infty} d(x_n, y_n) \text{ exists}$$

Proof. Done in class as part of the proof that $\operatorname{diam}(\operatorname{cl} Y) = \operatorname{diam} Y$. □

Proof. #1. (of theorem; this is the long but instructive proof) Let X be a metric space; assume X is not complete (or we are already done). Let C be the collection of all Cauchy sequences in X.

Let's define an equivalence relation on C, $\sim$:

Two sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ are equivalent if

$$\lim_{n\to\infty} d(x_n, y_n) = 0$$

This is an equivalence relation:

1. Reflexive: $d(x_n, x_n) = 0$ for all n
2. Symmetric: d is symmetric
3. Transitive: Let $(x_n) \sim (y_n)$, $(y_n) \sim (z_n)$.

   Then
   $$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, z_n) = 0$$
   Let $\varepsilon > 0$. Choose N so that $d(y_n, z_n), d(x_n, y_n) < \varepsilon/2$ for all $n \geq N$. □

Consider $(C, \sim)$, let Y be the set of all equivalence classes $[c]$ of C.
Let d_Y be such that $d_Y([c], [c']) = \lim_{n\to\infty} d_x(x_n, y_n)$ where $(x_n) \in [c]$ and $(y_n) \in [c']$.
This is well-defined:
Let $(x_n), (x'_n) \in [c]$, $(y_n), (y'_n) \in [c']$. We need to show that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$$

From class, we have the inequality:

$$\left| d(x_n, y_n) - d(x'_n, y'_n) \right| \leq d(x_n, x'_n) + d(y_n, y'_n) \to 0$$

□

Additionally, d_Y is a metric:

1. $d_Y([c], [c']) \geq 0$ obvious (limit of any sequence of positive reals is a non-negative real)
2. $d_Y([c], [c']) = 0 \iff [c] = [c']$ clear
3. symmetric
4. Triangle inequality: $\forall [c], [c'], [c''], d_Y([c], [c'']) \leq d_Y([c], [c']) + d_Y([c'], [c''])$ "also clear if you think about it for a minute"

Let $[c]_x$ denote the equivalence class that contains the sequence $x_n = x \ \forall x$, for some $x \in X$.
Define

$$f : X \to \tilde{Y}$$

where $\tilde{Y} = \{[c]_x : x \in X\}$. such that $f(x) = [c]_x$.

$$d_Y(f(x), f(y)) = d_Y([c]_x, [c]_y) = d_x(x, y)$$

Claim. $\mathrm{cl}(\tilde{Y}) = Y$.

Proof. Let $[c] \in Y$, and $(x_n) \in [c]$. $\exists N > 0$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$, $\forall n, m \geq N$.
Thus, $d(x_n, x_N) < \frac{\varepsilon}{2}$. $x_N \in X$.
Consider the equivalence class $[c]_{x_N} \in \tilde{Y}$. Then

$$d_Y([c], [c]_{x_N}) = \lim_{n\to\infty}(x_n, x_N) \leq \tfrac{\varepsilon}{2} < \varepsilon$$

□

Let $[c]^{(n)}$ be a sequence in Y that is Cauchy.

$$\implies \forall\varepsilon > 0, \exists N \text{ s.t. } \forall n, m \geq N, d_Y([c]^{(n)}, [c]^{(m)}) < \varepsilon$$

Then,

$$\lim_{j\to\infty} d(x_j^{(n)}, x_j^{(m)}) < \varepsilon$$

for representatives $x_j^{(n)} \in [c]^{(n)}$ and $x_j^{(m)} \in [c]^{(m)}$.
Then $\exists j(n)$ such that $\forall j \geq j(n)$,

$$d(x^n_{j(n)}, x_j^{(n)}) < \tfrac{1}{n}$$

We can assume wlog that $j(n) \to \infty$ as $n \to \infty$.
Let $y_n = x^{(n)}_{j(n)}$ and $y = (y_n)_{n=1}^{\infty}$.

Claim. y in X is Cauchy.

Proof. Let $\varepsilon > 0$. Choose N' such that $\frac{1}{N'} < \frac{\varepsilon}{3}$.
Choose N'' such that for all $n, m \geq N''$,

$$\lim_{j\to\infty} d(x_j^{(n)}, x_j^{(m)}) < \frac{\varepsilon}{4}$$

Then

$$d(y_n, y_m) = d(x^{(n)}_{j(n)}, x^{(m)}_{j(m)}) < d(x^{(n)}_{j(n)}, x_j^{(n)}) + d(x_j^{(n)}, x_j^{(m)}) + d(x_j^{(m)}, x^{(m)}_{j(m)})$$

Let's pick $j \geq \max(j(n), j(m))$ to obtain

$$d(x^{(n)}_{j(n)}, x_j^{(n)}) < \frac{1}{n}$$

$$d(x^{(n)}_{j(m)}, x_j^{(m)}) < \frac{1}{m}$$

If $m, n \geq \max\{N', N''\}$,

$$d(y_n, y_m) \leq \frac{1}{n} + \frac{\varepsilon}{3} + \frac{1}{m} \leq \frac{\varepsilon}{3} + \frac{2}{N'} \leq \varepsilon$$

□

Let $(y_n) \in [c]$.

Claim. (Last one):

$$\lim_{n\to\infty} d_y([c]^{(n)}, [c]) = 0$$

Proof.

$$\lim_{n\to\infty} d_y([c]^{(n)}, [c]) = \lim_{n\to\infty}\left(\lim_{j\to\infty} d(x_j^{(n)}, y_j)\right)$$

$$d(x_j^{(n)}, y_j) \leq d(x_j^{(n)}, x^{(n)}_{j(n)}) + \cancel{d(x^{(n)}_{j(n)}, y_j)}$$

$$\lim_{j\to\infty} d(x_j^{(n)}, x^{(n)}_{j(n)}) \leq \frac{1}{n}$$

For all $\varepsilon > 0$, $\exists J > 0$ such that $\forall j, j' \geq J$,

$$d(y_j, y_{j'}) < \varepsilon$$

So for $n \geq J$,

$$\lim_{j\to\infty} d(y_n, y_j) < \varepsilon$$

And thus,

$$\lim_{j\to\infty} d(x^{(n)}, y_j) < \frac{1}{n} + \varepsilon$$

So,

$$\limsup_{n\to\infty} \left(\lim_{j\to\infty} d(y_j^{(n)}, y_j) \right) = 0$$

Since the metric is always non-negative, we get

$$\lim_{n\to\infty} \left(\lim_{j\to\infty} d(y_j^{(n)}, y_j) \right) = 0$$

which is the same as

$$\lim_{n\to\infty} d_Y([c]^{(n)}, [c]) = 0$$

□

□

Exercise: If $(X, \|\cdot\|_X)$ is a normed vector space. Then if $(Y, \|\cdot\|_Y)$ is its completion, $(Y, \|\cdot\|_Y)$ is a Banach space.

Proof. #2. (of theorem; the short version) Let $C(X, \mathbb{R})$ be the space of all continuous bounded functions $f : X \to \mathbb{R}$, and let $\|f\| = \sup_{x\in X} |f(x)|$.

Remark. $(C(X, \mathbb{R}), \|\cdot\|)$ is a Banach space

Let $a \in X$ be given. And for each $x \in X$, set $f_x(z) = d_X(x, z) - d_X(a, z)$.

Claim. $f_x : X \to \mathbb{R}$ *is a bounded continuous function.*

Proof.

$$|f_x(z_1)-f_x(z_2)| \leq |d_X(x, z_1)-d_X(a, z_1)|+|d_x(x, z_2)-d_x(a, z_2)| \leq d_X(z_1, z_2)+d_X(z_1, z_2) = 2d_X(z_1, z_2)$$

So $f_x(z)$ is uniformly continuous.

$$|f_x(z)| = |d_X(x, z) - d_X(a, z)| \leq d_X(a, x) < \infty$$

So $f_x(z) \in C(X, \mathbb{R})$.

□

Let $\tilde{Y} = \{f_x : x \in X\}$.
Let $F : X \to \tilde{Y}$ with $F(x) = f_x$.

$$\begin{aligned}\|f_x - f_y\| &= \sup_{z\in X} |f_x(z) - f_y(z)| = \sup_{z\in X} |d_X(x, z) - d_X(a, z) - d_X(a, z) - d_X(y, z)| \\ &= \sup_{z\in X} |d_X(x, z) - d_X(y, z)| \\ &\geq d(x, y) \\ &\leq d(x, y)\end{aligned}$$

□

§16 Tuesday, October 1, 2013

§16.1 Extension by uniform continuity

Theorem 16.1 ***(Extension by uniform continuity)*** *Let (X, d_X) and (Y, d_Y) be metric spaces and suppose that Y is complete.*

Let $S \subset X$ be a dense set (this means $\operatorname{cl} S = X$) and $f : S \to Y$ a uniformly continuous function; this means

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in S)(d_X(x,) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$$

Then there exists a unique uniformly continuous function

$$\bar{f} : X \to Y$$

§16.1.1 Proof

Uniqueness: Let $\bar{f}$ and $\bar{g}$ be two extensions satisfying the conditions of the theorem. Let $x \in X$ and let $(x_n)_{n=1}^{\infty}$ be a sequence in S converging to x. (We know such a sequence exists because $\operatorname{cl} S = X$).

Then,

$$\bar{f}(x_n) = f(x_n) = \bar{g}(x_n) \tag{$\star$}$$

Since $\bar{f}$ and $\bar{g}$ are continuous,

$$\bar{f}(x) = \lim_{n\to\infty} \bar{f}(x_n)$$

and

$$\bar{g}(x) = \lim_{n\to\infty} \bar{g}(x_n)$$

Then $(\star)$ implies that $\bar{f}(x) = \bar{g}(x)$.

Existence:

Claim. *Let (x_n) be a sequence in S converging to a point $x \in X$. Then $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in Y.*

Proof. Let $\varepsilon > 0$. By uniform continuity of f on S, there exists $\delta > 0$ such that for all $x, y \in S$,

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Note x_n is a converging sequence, hence it is Cauchy. So we can find $N > 0$ such that for all $m, n \geq N$, $d_X(x_n, x_m) < \delta$.

But then,

$$d_Y(f(x_n), f(x_m)) < \varepsilon$$

□

Back to the proof: How to define our extension $\bar{f}$:

If $x \in X$, let (x_n) be a sequence in S converging to x. Then $f(x_n)$ is Cauchy in Y, and since Y is complete,

$$\lim_{n\to\infty} f(x_n) \in Y$$

So we set $\bar{f}(x) = \lim_{n\to\infty} f(x_n)$.

First, we need to check that $\bar{f}$ is well defined. That is, if (x_n) and (y_n) are two sequences in S converging to $x \in X$, then we must have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$$

Define a sequence z_n by $z_{2n-1} = x_n$, and $z_{2n} = y_n$. (i.e. $z = (x_1, y_1, x_2, y_2, \ldots)$).

Note that z_n also converges to x and obviously, $z_n \in S$.

$\lim_{n\to\infty} f(z_n)$ exists by the claim But then

$$\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$$

So $\bar{f}$ is well-defined.

Now if $x \in S$, we can take for (x_n) the constant sequence $x_n = x$. Then $f(x_n) = f(x)$ and $\bar{f}(x) = \lim_{n\to\infty} f(x_n) = f(x)$. So $\bar{f}$ coincides with f on S.

It remains to check that $\bar{f}$ is uniformly continuous on X. So we want to show

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d_X(x,y) < \delta \implies d_Y(\bar{f}(x), \bar{f}(y)) < \varepsilon)$$

Fix $\varepsilon > 0$. We will choose δ so that for all $x, y \in S$,

$$d_X(x,y) < 2\delta \implies d_Y(f(x), f(y)) < \frac{\varepsilon}{3}$$

(This is possible because f is uniformly continuous on S). The claim is that this δ does the job. Let $x, y \in X$ be such that $d_X(x,y) < \delta$.

Choose sequences (x_n) and (y_n) in S such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then $\exists N' > 0$ such that for all $n \geq N'$,

$$d_X(x_n, x) < \frac{\delta}{2} \text{ and } d_X(y_n, y) < \frac{\delta}{2}$$

Since $\lim_{n\to\infty} f(x_n) = \bar{f}(x)$ and $\lim_{n\to\infty} f(y_n) = \bar{f}(y)$, $\exists N'' > 0$ such that for $n \geq N''$,

$$d_Y(f(x_n), \bar{f}(x)) < \frac{\varepsilon}{3} \text{ and } d_Y(f(y_n), \bar{f}(y)) < \frac{\varepsilon}{3}$$

Take any $n \geq \max(N', N'')$. Then

$$d_X(x_n, y_n) \leq \underbrace{d_X(x_n, x)}_{\delta/2} + \underbrace{d_X(x, y)}_{\delta} + \underbrace{d_X(y, y_n)}_{\delta/2} < 2\delta$$

and

$$d_Y(\bar{f}(x), \bar{f}(y)) \leq d_Y(\bar{f}(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(f(y_n), \bar{f}(y))$$

these are each less than $\varepsilon/3$, so $d_Y(\bar{f}(x), \bar{f}(y)) < \varepsilon$.

□

§16.2 Exercise 45:

Remark. Let (X_k, d_k), $k = 1, \ldots, n$ be complete metric spaces. Then their product $X = \prod_{k=1}^n X_k$ is also complete. In particular, $\mathbb{R}^n$ is complete. (This is left as an exercise).

Let (X_k, d_k), $k = 1, 2, \ldots$ be a sequence of metric spaces and

$$X = \prod_{k=1}^{\infty} X_k = \{x = (x_k)_{k=1}^{\infty} : x_k \in X_k\}$$

and

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

If each (X_k, d_k) is complete, then the product (X, d) is complete.

Proof. Let $x^{(n)} = (x_k^{(n)})_{k=1}^{\infty}$, $n = 1, 2, \ldots$ be a Cauchy sequence in (X, d). We need to show that it converges. We have shown that $x^{(n)}$ converges if and only if for all k, the sequence

$$x_k^{(n)}, n = 1, 2, \ldots \text{ converges in } (X_k, d_k)$$

Since (X_k, d_k) is complete, it suffices to show that $x_k^{(n)}$ is Cauchy.

So we need to show that if $x^{(n)}$ is Cauchy in (X,d), then for any k, the sequence $x_k^{(n)}$ is Cauchy in (X,d). Fix k. Choose $\varepsilon > 0$. Let $\varepsilon' > 0$ be such that

$$0 < \frac{2^k\varepsilon'}{1-2^k\varepsilon'} < \varepsilon$$

Since $x^{(n)}$ is Cauchy, we know that there $\exists N > 0$ such that $\forall m,n \geq N$,

$$d(x^{(n)},x^{(m)}) = \sum_{j=1}^{\infty} 2^{-j}\frac{d_j(x_j^{(n)},x_j^{(m)})}{1+d_j(x_j^{(n)},x_j^{(m)})} < \varepsilon'$$

In particular,

$$2^{-k}\frac{d_j(x_j^{(n)},x_j^{(m)})}{1+d_j(x_j^{(n)},x_j^{(m)})} < \varepsilon'$$

Then

$$d_k(x_k^{(n)},x_k^{(m)}) < 2^k\varepsilon' + 2^k\varepsilon' d_k(x_k^{(n)},x_k^{(m)})$$

so

$$d_k(x_k^{(n)},x_k^{(m)})(1-2^k\varepsilon') < 2^k\varepsilon'$$

and thus

$$d_k(x_k^{(n)},x_k^{(m)}) < \frac{2^k\varepsilon'}{1-2^k\varepsilon'} < \varepsilon$$

for all $n,m \geq N$.

So $x_k^{(n)}$, $n = 1,2,\ldots$, is Cauchy and we are done. □

§16.3 9.5: Compact Metric Spaces

Recall: Bolzano Wierstrauss Theorem: $[a,b]$ closed and bounded interval, $(x_n)_{n=1}^{\infty}$ with $x_n \in [a,b]$. Then there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k}$ exists.

§17 Thursday, October 3, 2013

§17.1 9.5: Compact Metric Spaces

Definition 17.1 Let X be a set and $Y \subset X$. A family of sets $\{O_\alpha\}_{\alpha\in I}$, $O_\alpha \subset X$ is called a cover of Y if

$$Y \subset \bigcup_{\alpha\in I} O_\alpha$$

A cover $\{O_\alpha\}_{\alpha\in I}$ is said to have a finite subcover if there are finitely many indicies $\alpha_1,\ldots,\alpha_n \in I$ such that

$$Y \subset O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$$

If X is a metric space, a cover $\{O_\alpha\}_{\alpha\in I}$ is called an open cover if each O_α is an open set in X.

Definition 17.2 A metric space X is called compact if every open cover of X has a finite subcover.

A subset $K \subset X$ is called compact if K is compact considered as a metric subspace of X.

Remark. A subset $K \subset X$ is compact if and only if every open cover of K in X has a finite subcover.

Proof. Suppose that K is compact. Then K as a metric subspace of X is compact. Let $\{O_\alpha\}$ be an open cover of K in X.

$$K \subset \bigcup_{\alpha\in I} O_\alpha$$

where O_α is open in X.

$$K = \bigcup_{\alpha \in I} O_\alpha \bigcap K = \bigcup_{\alpha \in I} V_\alpha$$

for sets V_α open in K, so $\{V_\alpha\}$ is an open cover of K (in K). Since K is compact, $\{V_\alpha\}$ has a finite subcover,

$$K = V_{\alpha_1} \cup \ldots \cup V_{\alpha_n} = K \bigcap (O_{\alpha_1} \cup \ldots \cup O_{\alpha_n})$$

So

$$K \subset O_{\alpha_1} \cup \ldots \cup o_{\alpha_n}$$

Conversely, suppose now that every open cover of K in X has a finite subcover. Let $\{V_\alpha\}$ be an open cover of K in (K, d). So, each V_α is open in (K, d), and

$$K = \bigcup_{\alpha \in I} V_\alpha$$

We have shown that

$$V_\alpha = O_\alpha \bigcap K$$

where O_α is open in K.

Then

$$K = \bigcup_{\alpha \in I} K \cap O_\alpha = K \cap \left(\bigcup_{\alpha \in I} O_\alpha \right) \subset \bigcup_{\alpha \in I} O_\alpha$$

So $O_{\alpha \alpha \in I}$ is an open cover of K in X.

So we can pick a finite subcover $K \subset O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$. But then

$$K = K \cap (O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}) = (K \cap O_{\alpha_1} \cup \ldots \cup (K \cap O_{\alpha_n}) = V_{\alpha_1} \cup \ldots V_{\alpha_n}$$

So $V_{\alpha_1}, \ldots, V_{\alpha_n}$ is a finite subcover. □

Definition 17.3 A family of sets $\{F_\alpha\}_{\alpha \in I}$ in X is said to have the finite intersection property if for any finite subfamily $F_{\alpha_1} \ldots, F_{\alpha_n}$,

$$F_{\alpha_1} \cap \ldots \cap F_{\alpha_n} \neq \varnothing$$

Proposition 17.4 Let X be a metric space. T.F.A.E.

1. X is compact
2. For any family $\{F_{\alpha_n}\}$ of closed sets in X with the finite intersection property,

$$\bigcap_{\alpha \in I} F_\alpha \neq \varnothing$$

Proof. (1) $\implies$ (2). We argue by contradiction. Suppose X is compact, but (2) fails.

This means $\exists \{F_\alpha\}_{\alpha \in I}$ a family of closed sets with finite intersection property with

$$\bigcap_{\alpha \in I} F_\alpha = \varnothing$$

Then

$$\bigcup_{\alpha \in I} F_\alpha^c = X$$

Since F_α is closed, $\{F_\alpha^c\}$ is an open cover of X. Since X is compact, there exists a finite subcover

$$X = F_{\alpha_1}^c \cup \ldots \cup F_{\alpha_n}^c$$

Then

$$\varnothing = X^c - \left(F_{\alpha_1}^c \cup \ldots \cup F_{\alpha_n}^c\right)^c = (F_{\alpha_1}^c)^c \cap \ldots \cap (F_{\alpha_n}^c)^c = F_{\alpha_1} \cap \ldots \cap F_{\alpha_n}$$

This contradicts the fact that $\{F_\alpha\}$ has the finite intersection property.

To show that (2) $\implies$ (1), Let $\{O_\alpha\}_{\alpha\in I}$ be an open cover of X. Then

$$X = \bigcup_{\alpha\in I} O_\alpha$$

Then $\varnothing = \bigcap_{\alpha\in I} O_\alpha^c$. So the family $\{O_\alpha^c\}_{\alpha\in i}$ of closed sets does not have the finite intersection property (by (2)).

Hence, there are finitely many $\alpha_1, \ldots, \alpha_n \in I$ such that

$$\varnothing = O_{\alpha_1}^c \cap \ldots \cap O_{\alpha_n}^c$$

and then

$$X = \left(O_{\alpha_1}^c \cap \ldots \cap O_{\alpha_n}^c\right)^c = O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$$

So the cover $\{O_\alpha\}$ has a finite subcover, and X is compact. □

Definition 17.5 Let X be a metric space. Then X is called totally bounded if for any $\varepsilon > 0$, X can be covered by a finite set of balls of radius ε. A subset $Y \subset X$ is called totally bounded if Y is totally bounded as a metric subspace of X.

Remark. X is totally bounded if for any $\varepsilon > 0$ there are $x_1, \ldots, x_n \in X$ (n- depends on ε) such that

$$X = B(x_1, \varepsilon) \cup \ldots \cup B(x_n, \varepsilon)$$

$Y \subset X$ is totally bounded if for any $\varepsilon > 0$ there are $x_1, \ldots, x_n \in Y$ such that $Y \subset B(x, \varepsilon) \cup \ldots \cup B(x_n, \varepsilon)$

Fact. If X is totally bounded, then X is bounded. Take $\varepsilon = 1$ and write

$$X = B(x_1, 1) \cup \ldots \cup B(x_n, 1)$$

Let $x, y \in X$. Then $x \in B(x_k, 1)$ and $y \in B(x_j, 1)$.

$$d(x, y) \le d(x, x_k) + d(x_j, x_k) + d(x_j, y) \le 2 + \max_{1\le \ell,\ell' \le n} d(x_\ell, x_{\ell'}) =: M$$

So, $\forall x, y \in X$, $d(x, y) \le M$. So $\operatorname{diam}(X) < \infty$ and X is bounded.

Remark. Very important: converse is not true in general.

Example 17.6 Normed space $(C([0,1]), \|\cdot\|_2)$.

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 \,\mathrm{d}x\right)^{1/2}$$

Choose the unit ball in this normed space:

$$B = \{f \in C([0,1]) : \|f\|_2 \le 1\}$$

Clearly, bounded.

Let $f, g \in B$. Then

$$d(f, g) = \|f - g\|_2 \le \|f\|_2 + \|g\|_2 \le 2$$

So $\operatorname{diam}(B) \le 2$.

Let $f_n(x) = \cos(2\pi n x)$, $n = 0, 1, \ldots$.

We have that $\|f_n\|_2 \le 1$.

$$\begin{aligned}\|f_n - f_m\|_2 &= \int_0^1 |f_n - f_m|^2 \,\mathrm{d}x = \int_0^1 |\cos(2\pi n x) - \cos(2\pi m x)|^2 \,\mathrm{d}x \\ &= \int_0^1 \cos^2(2\pi n x) \,\mathrm{d}x + \int_0^1 \cos^2(2\pi m x) \,\mathrm{d}x - 2\int_0^1 \cos(2\pi m x)\cos(2\pi m x) \,\mathrm{d}x\end{aligned}$$

Then use that $\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$ and that $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))$. We get that

$$\|f_n - f_m\|_2 = 1 \text{ if } n \neq m$$

If B was totally bounded, we could find $g_1, \ldots, g_\ell$ such that

$$B \subset B(g_1, \frac{1}{3}) \cup \ldots \cup B(g_\ell, \frac{1}{3})$$

$\{f_n\}_{n=1}^{\infty}$ is in B. So there are $n \neq m$ such that

$$f_n, f_m \in B(g, \frac{1}{3})$$

But then

$$1 = \|f_n - f_m\|_2 \leq \|f_n - g\|_2 + \|g - f_m\|_2 < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Contradiction: B is not totally bounded.

Theorem 17.7 *Let $(X, \|\cdot\|)$ be a normed space in which the closed ball*

$$B = \{x \in X : \|X\| \leq 1\}$$

is totally bounded. Then $\dim X < \infty$.

Theorem 17.8 *If $(X, \|\cdot\|)$ is a normed space and $\dim X < \infty$ then $Y \subset X$ is bounded if and only if it is totally bounded.*

§18 Friday, October 4, 2013

§18.1 Characterization of compactness

Definition 18.1 A metric space X is called sequentially compact if any sequence $(x_n)_{n=1}^{\infty}$ in X has a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that (x_{n_k}) converges to an element of X as $k \to \infty$.

Theorem 18.2 ***(Characterization of compactness)*** *Let X be a metric space. T.F.A.E.:*

1. *X is complete and totally bounded*
2. *X is compact*
3. *X is sequentially compact*

We will split the proof into three propositions.

Proposition 18.3 $((1) \implies (2))$ If X is totally bounded and complete, then X is compact.

Proof. Suppose X is totally bounded and complete, but not compact.

Then, there exists an open cover $\{O_\alpha\}$ of X that does not have a finite subcover. We will construct a sequence $F_1, F_2, \ldots, F_n, \ldots$ of closed sets in X as follows:

Step 1: Since X is totally bounded, there are finitely many points $x_1^{(1)} \ldots, x_n^{(1)}$ such that

$$X = B(x_1^{(1)}, 1/2) \cup \ldots \cup B(x_n^{(1)}, 1/2)$$

there must exists k with $1 \leq k \leq n$ such that the ball $B(x_k^{(1)}, 1/2)$ is *not* covered by any finite subcollection of $\{O_\alpha\}$ (otherwise we would have found a finite subcover).

Then we set

$$F_1 = \mathrm{cl}(B(x_k^{(1)}, 1/2)$$

Note $\mathrm{diam}(F_1) \leq 1$.

Step 2: We construct F_2 by using F_1. Again, since X is totally bounded, there are $x_1^{(2)}, x_2^{(2)}, \ldots, x_{n_2}^{(2)}$ such that

$$X = B(x_1^{(2)}, 1/4) \cup \ldots \cup B(x_{n_2}^{(2)}, 1/4)$$

Then

$$F_1 = F_1 \cap X = \bigcup_{j=1}^{n} F_1 \cap B(x_j^{(2)}, 1/4)$$

Now there must exist k, $1 \leq k \leq n_2$ such that

$$F_1 \bigcap B(x_k^{(2)}, 1/4)$$

is not covered by any finite subcollection of $\{O_\alpha\}$, because we know that F_1 cannot be covered by any finite subcollection.

Set

$$F_2 = \mathrm{cl}(F_1 \cap B(X_k^{(2)}, 1/4))$$

So we have F_1, F_2 both closed, $F_2 \subset F_1$, $\operatorname{diam} F_1 \leq 1$, $\operatorname{diam} F_2 \leq \frac{1}{2}$, and both F_1 and F_2 cannot be covered by any finite subcollection of $\{O_\alpha\}$.

Suppose that we have constructed $F_1, \ldots, F_{N-1}$ such that

$$F_1 \supset F_2 \supset \ldots \supset F_{N-1}$$

and $\operatorname{diam} F_n \leq \frac{1}{n}$, and each F_n cannot be covered by any finite subcollection of $\{O_\alpha\}_{\alpha \in I}$, for all $1 \leq n \leq N-1$.

Step N: F_N is constructed as follows:

$$X = B(x_1^{(N)}, \frac{1}{2N}) \cup \ldots \cup B(x_{n_N}^{(N)}, \frac{1}{2N})$$

Then there must exist k, $1 \leq k \leq n_N$ such that

$$F_{N_1} \cap B(x_k^{(N)}, \frac{1}{2N})$$

cannot be covered by any finite subcollection of $\{O_\alpha\}$. Then set

$$F_N = \mathrm{cl}(F_{N-1} \cap B\left(x_k^{(N)}, \frac{1}{2N}\right)$$

Then F_N is closed, $F_{N-1} \supset F_N$, F_N cannot be covered by any finite subcollection of $\{O_\alpha\}$ and $\operatorname{diam} F_N \leq \frac{1}{N}$.

So we have a sequence F_N, $N = 1, 2, \ldots$ of closed sets satisfying the above properties.

So far we have used that X is totally bounded, but not that X is complete. Now we use that X is complete to apply Cantor Intersection Theorem: $\exists x_0 \in X$ such that

$$\bigcap_{N=1}^{\infty} F_N = \{x_0\}$$

Now, since $\{O_\alpha\}_{\alpha \in I}$ covers X there exists an $\alpha_0 \in I$ such that

$$x_0 \in O_{\alpha_0}$$

Since O_{α_0} is open, there exists $r > 0$ such that

$$B(x_0, r) \subset O_{\alpha_0}$$

Choose now N_0 such that $\frac{1}{N_0} < r$. Then we know that $x_0 \in F_{N_0}$ and $\operatorname{diam} F_{N_0} < \frac{1}{N_0} < r$.

So if $y \in F_{N_0}$, then

$$d(x_0, y) \leq \operatorname{diam}(F_{N_0}) < r$$

So,

$$F_{N_0} \subset B(x_0, r) \subset O_{\alpha_0}$$

So F_{N_0} is covered by O_{α_0} and F_{N_0} has a finite subcover in $\{O_\alpha\}$. But by construction, F_{N_0} does not have a finite subcover.

□

Proposition 18.4 ((2) $\implies$ (3)). If X is compact, then X is sequentially compact.

Proof. Let $(x_n)_{n=1}^\infty$ be a sequence in X.

Define $F_n = \operatorname{cl}\{x_k : k \geq n\}$. So F_n is closed, $F_{N+1} \subset F_N$. (You cannot apply Cantor intersection theorem, because we don't know that X is complete).

The family $\{F_n\}_{n=1}^\infty$ has the finite intersection property:

$$F_{n_1} \cap \ldots \cap F_{n_k} \ni x_\ell$$

where $x_\ell = \max(n_1, \ldots, n_k)$. Since X is compact,

$$\bigcap_{n=1}^\infty F_n \neq \varnothing$$

Pick

$$x_0 \in \bigcap_{n=1}^\infty F_n$$

$$x_0 \in F_1 = \operatorname{cl}(\{x_k : k \geq 1\})$$

So, by definition of closure point,

$$B(x_0, 1) \cap \{x_k : x \geq 1\} \neq \varnothing$$

Pick x_{n_1} in the set $B(x_0, 1) \cap \{x_k : x \geq 1\}$. Then $d(x_{n_1}, x_0) < 1$.

Then $x_0 \in F_{n_1+1} = \operatorname{cl}\{x_k : k \geq n_1 + 1\}$. So,

$$B(x_0, 1/2) \cap \{x_k : k \geq n_1 + 1\} \neq \varnothing$$

Pick x_{n_2} in this set. $n_2 > n_1$, $d(x_{n_2}, x_0) < \frac{1}{2}$. Continue inductively.

$x_{n_k} \in F_{n_{k-1}+1}$ such that $d(x_{n_k}, x_0) < \frac{1}{k}$.

Then (x_{n_k}) is a subsequence of (x_n) and

$$d(x_{n_k}, x_0) < \frac{1}{k}$$

Thus,

$$\lim_{k \to \infty} d(x_{n_k}, x_0) = 0 \iff (x_{n_k}) \to x_0$$

□

Proposition 18.5 ((3) $\implies$ (1)). If X is sequentially compact, then X is complete and totally bounded.

Proof. Let (x_n) be a Cauchy sequence in X. Since X is sequentially compact, (x_n) has a convergent subsequence (x_{n_k}). Then (x_n) converges, because we have shown that a Cauchy sequence with convergent subsequence has to converge. Thus, X is complete.

To prove that X is totally bounded, we argue by contradiction. Suppose it is not. Then $\exists \varepsilon_0 > 0$ such that X is not covered by a finite collection of balls of radius ε_0.

Pick $x_1 \in X$. Then $X \neq B(x_1, \varepsilon_0)$. So $\exists x_2 \notin B(x_1, \varepsilon_0)$, i.e. $d(x_2, x_1) \geq \varepsilon_0$.

Then $X \neq B(x_1, \varepsilon_0) \cup B(x_2, \varepsilon_0)$.

So $\exists x_3 \notin B(x_1, \varepsilon_0) \cup B(x_2, \varepsilon_0)$, so $d(x_3, x_2) \geq \varepsilon_0$ and $d(x_3, x_1) \geq \varepsilon_0$.

Continue inductively. We construct in this way a sequence $(x_n) \in X$ such that $\forall m \neq n$, $d(x_m, x_n) \geq \varepsilon_0$. Since X is sequentially compact, (x_n) has a convergent subsequence (x_{n_k}). But $d(x_{n_k}, x_{n_j}) \geq \varepsilon_0$ if $k \neq j$. So (x_{n_k}) is not Cauchy, and so cannot be convergent. This is a contradiction, so X must be totally bounded.

□

§19 Friday, October 4, 2013: Tutorial IV

§19.1 $\boxed{\ell^p\text{–spaces}}$

Recall: $\|\cdot\|_p$ on $\mathbb{R}^n$.

$$x = (x_i)_{i=1}^n \qquad \|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

and

$$\|x\|_\infty = \max_{1\leq k\leq n} |x_k|$$

$$\mathbb{R}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{R}\}$$

$$\|f\|_p = \left(\sum_{k=1}^\infty |f_k|^p\right)^{1/p}$$

$$\|f\|_\infty = \sup_{k\geq 1} |f_k|$$

A priori, this is not a norm on $\mathbb{R}^{\mathbb{N}}$; $f_k = k$ leads to $\|f\|_p = \infty$ for all $1 \leq p \leq \infty$.

$$\ell^p(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_p < \infty\} = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^\infty |x_k|^p < \infty\}$$

And, similarly,

$$\ell^\infty(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sup_{k\geq 1} |x_k| < \infty$$

Easy to check that $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{N})$, for $1 \leq p \leq \infty$.

Any countably infinite set is in bijection to $\mathbb{N}$, so we will only consider $\ell^p(\mathbb{N})$.

Fact. If $p \leq q$, then $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$.

Proof. Assume $p \leq q$.

$$\begin{aligned} x \in \ell^p(\mathbb{N}) &\iff \sum_{k=1}^\infty |x_k|^p < \infty \\ &\implies \lim_{k\to\infty} |x_k| = 0 \\ &\implies \exists N > 0 \text{ s.t. } \forall k \geq N, |x_k| < 1 \\ |x_k| < 1 &\implies |x_k|^p \geq |x_k|^q \, \forall k \geq N \\ \sum_{k=1}^\infty |x_k|^q &= \underbrace{\sum_{k=1}^N |x_k|^q}_{<\infty} + \underbrace{\sum_{k=N+1}^\infty |x_k|^p}_{<\infty} \end{aligned}$$

So $x \in \ell^q(\mathbb{N})$.

It's left as an exercise to show that if $p < q$, then $\ell^p(\mathbb{N}) \subsetneq \ell^q(\mathbb{N})$.

□

§19.2 Holder and Minkowski inequalities on ℓ^p spaces

Minkowksi inequality Let $x, y \in \ell^p(\mathbb{N})$, for $1 \leq p < \infty$.

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Holder's inequality If $x \in \ell^p$, $y \in \ell^q$, and $\frac{1}{p} + \frac{1}{q} = 1$, then $xy \in \ell^1(\mathbb{N})$ and

$$\|xy\|_1 \leq \|x\|_p \|y\|_q$$

§19.3 ℓ^p is Complete

Theorem 19.1 *$\ell^p(\mathbb{N})$ is complete* $(1 \leq p \leq \infty)$.

§19.3.1 Proof

($p = \infty$ as an exercise). Let $(a^k)_{k=1}^\infty$ be Cauchy in ℓ^p.

$$a^k = (a_i^k)_{i=1}^\infty a_i^k \in \mathbb{R}$$

a^k Cauchy $\iff \forall \varepsilon > 0, \exists K$ s.t. $\forall m, n \geq k$,

$$\|a^m - a^n\|_p < \varepsilon$$

Which is true if and only if

$$\sum_{i=1}^\infty |a_i^m - a_i^n|^p < \varepsilon^p$$

and we have that

$$|a_{i_0}^m - a_{i_0}^n|^p \leq \sum_{i=1}^\infty |a_i^m - a_i^n|^p < \varepsilon^4 \implies |a_{i_0}^m - a_{i_0}^n| < \varepsilon$$

for fixed $i_0 \in \mathbb{N}$.

So $(a_{i_0}^k)_{k=1}^\infty$ is Cauchy.

We have that

$$a_i^k \xrightarrow{k\to\infty} a_i \in \mathbb{R}$$

for all $i \in \mathbb{N}$.

Let $a = (a_i)_{i=1}^\infty$ where $a_i = \lim_{k\to\infty} a_i^k$.

We know that $a_1^k \to a_1$. Choose k_1 such that $\forall k \geq k_1$, we have $|a_1^k - a_1| < \frac{1}{2} = 2^{-1}$.

Since $a_i^k \to a_i$, choose $k_i \geq k_{i-1} \in \mathbb{N}$ such that $\forall k \geq k_i$, and $j \leq i$,

$$|a_j^k - a_j| < \frac{1}{2^i}$$

For each $j \in \mathbb{N}$, $k \geq k_j$, $n \leq j$, we have

$$|a_n^k - a_n| \leq 2^{-j}$$

In particular,

$$|a_n^{k_j} - a_n| < 2^{-j}$$

for $j \geq n$.

Claim. *$a = (a_n)_{n=1}^\infty \in \ell^p(\mathbb{N})$. Need to show $\sum_{k=1}^\infty |a_k|^p < \infty$.*

Proof. Fix $N \in \mathbb{N}$.

$$a_n = (a_n - a_n^{k_N}) + a_n^{k_N}$$

$$\left(\sum_{k=1}^{N} |a_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |a_n - a_n^{k_N}|^p\right)^{1/p} + \underbrace{\left(\sum_{k=1}^{N} |a_n^{k_N}|^p\right)^{1/p}}_{\leq \|a_n^{k_N}\|_p < \infty}$$

Recall that (a^k) is Cauchy in ℓ^p (by assumption), so bounded: $\exists R$ such that $\|a^{k_N}\|_p < R$ for all N.

Then,

$$\left(\sum_{n=1}^{N} |a_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{N} (2^{-N})^p\right)^{1/p} + R = \left(N2^{-Np}\right)^{1/p} + R = 2^{-N}(N)^{1/p} + R \leq R + S$$

$2^{-N}(N)^{1/p} \to 0$ as $N \to \infty$, so bounded above by S for all N.

Then we have $a \in \ell^p(\mathbb{N})$, as the limit exists. □

Claim. $\|a^k - a\|_p \to 0$ *as* $k \to \infty$.

Proof. Let $\varepsilon > 0$. Since $a \in \ell^p$, we have $\exists N_1 \in \mathbb{N}$ such that $\sum_{n=N_1}^{\infty} |a_n|^p < \varepsilon^p$.

In addition, (a^k) is Cauchy in ℓ^p, so $\exists N_2$ such that $\forall k, m \geq N_2$,

$$\|a^k - a^m\|_p < \varepsilon$$

Let $N = \max(N_1, N_2)$. Thus,

$$\sum_{n=N}^{\infty} |a_n|^p < \varepsilon^p \text{ and } \|a^N - a^k\|_p < \varepsilon, \forall k \geq N$$

a^N is Cauchy, so take N' large enough so that

$$\sum_{n=N'}^{\infty} |a_n^N|^p < \varepsilon^p$$

WLOG, take $N' \geq N$.

For each n, $a_n^k \to a_n$, so choose K_1 so that

$$|a_1^k - a_1| < \frac{\varepsilon^p}{N'}$$

for $k \geq K_1$. Similarly, choose K_i with $1 \leq i \leq N$ such that

$$|a_i^k - a_i| < \frac{\varepsilon^p}{N'}$$

and choose $K = \max_i K_i$.

Then

$$|a_i^k - a_n| < \frac{\varepsilon^p}{N'} \tag{$\star$}$$

for $k \geq K$, $n \leq N'$.

Again, WLOG, $K \geq N'$.

For any $k \geq K$,

$$(a_n^k - a_n)_{n=1}^{\infty} = (a_n^k - a_n)_{n=1}^{N'-1} + (a_n^k - a_n)_{n=N'}^{\infty}$$

aside interpret the first sequence as being infinite but zero after $N'-1$, and the second as starting at 1 but zero before N', so that the sum makes sense

$$\|a^k - a\|_p \leq \left(\sum_{n=1}^{N'-1} |a_n^k - a_n|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |a_n^k - a_n|^p\right)^{1/p}$$

Then by $(\star)$,

$$\left(\sum_{n=1}^{N'-1} |a_n^k - a_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{N'-1} \left(\frac{\varepsilon^p}{N'}\right)\right)^{1/p} = \left(\frac{N'-1}{N'}\right)^{1/p} \varepsilon$$

Then, by use of triangle and Minkowski,

$$\left(\sum_{n=N'}^{\infty} |a_n^k - a_n|^p\right)^{1/p} \leq \left(\sum_{n=N'}^{\infty} |a_n^k|^p\right)^{1/p} + \underbrace{\left(\sum_{n=N'}^{\infty} |a_n|^p\right)^{1/p}}_{<\varepsilon}$$

$$\|a^k - a\|_p \leq 4\varepsilon$$

□

§20 Tuesday, October 8, 2013

Theorem 18.2 ***(Characterization of compactness)*** *Let X be a metric space. T.F.A.E.:*

1. *X is complete and totally bounded*
2. *X is compact*
3. *X is sequentially compact*

§20.1 Two basic facts

Proposition 20.1 Let X be a metric space and $Y \subset X$ a compact subset. Then Y is closed and bounded.

Proof. Y is totally bounded so Y is bounded. To show that Y is closed, let $x \in \operatorname{cl} Y$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in Y that converges to x,

$$\lim_{n\to\infty} d(x_n, x) = 0 \tag{$\star$}$$

Since Y is compact, there exists a subsequence (x_{n_k}) that converges to an element of Y (compact $\iff$ sequentially compact).

Hence, there is $y \in Y$ and a subsequence (x_{n_k}) such that

$$\lim_{k\to\infty} d(x_{n_k}, y) = 0$$

But from $(\star)$ we have

$$\lim_{k\to\infty} d(x_{n_k}, x) = 0$$

So $x = y$ and this means that $x \in Y$. Hence, Y is closed.

□

Proposition 20.2 (Exercise) Let X be a compact metric space and $Y \subset X$. T.F.A.E.

1. Y is compact
2. Y is closed

Remark. Recall similar statement for completeness.

§20.2 Compactness and continuous functions

Proposition 20.3 Let (X, d_x) and (Y, d_y) be metric spaces and $f : X \to Y$ a continuous function. Let X be compact. Then the image of X by f, that is, the set

$$f(X) = \{f(x) : x \in X\}$$

is a compact subset of Y.

Proof. Let $\{O_\alpha\}_{\alpha \in I}$ be an open cover of $f(X)$.

$$f(X) \subset \bigcup_{\alpha \in I} O_\alpha$$

Then

$$X \subset f^{-1}\left(\bigcup_{\alpha \in I} O_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$$

and (since $f : X \to Y$)

$$X = \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$$

f is continuous, O_α is open in Y, so $f^{-1}(O_\alpha)$ is open in X.

So, $\{f^{-1}(O_\alpha)\}_{\alpha \in I}$ is an open cover of X. Since X is compact, this cover has a finite subcover. Thus, $\exists \alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that

$$X = f^{-1}(O_{\alpha_1}) \cup \ldots \cup f^{-1}(O_{\alpha_n})$$

But then,

$$f(X) = \bigcup_{k=1}^{n} f\left(f^{-1}(O_{\alpha_k})\right) \subset \bigcup_{k=1}^{n} O_{\alpha_k}$$

So we have extracted the finite subcover, so $f(X)$ is compact. □

Exercise 20.4 Properties of f, f^{-1}, $\Big($for example, we have used that $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$ $\Big)$. Make sure you know all of them.

Exercise 20.5 Give another proof of the last proposition by showing that $f(X)$ is sequentially compact.

Theorem 20.6 ***(Extreme Value Theorem)*** *Let X be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. Then f achieves its minimum and maximum value on X.*

Remark. For the converse statement, refer to the book.

Proof. $f(X)$ is compact subset of $\mathbb{R}$, and hence closed and bounded. Let

$$M = \sup f(X) = \sup\{f(x) : x \in X\}$$

and

$$m = \inf f(X)$$

Then, obviously, M and m are finite since $f(X)$ is bounded. Then the claim is that there exist $x_0, y_0 \in X$ such that

$$f(x_0) = M \text{ and } f(y_0) = m$$

Then $f(x_0) \geq f(x)\ \forall x \in X$ and f achieves its maximum value at x_0. Similarly for y_0.

We will only deal with M (maximum value); the other case follows from the first by considering $-f$.

Suppose that the statement is not true: for all $x \in X$, $f(x) < M = \sup f(X)$.

Let $n > 0$ and look at $M - \frac{1}{n}$. This is not an upper bound for $f(X)$, since M, as the supremum, is the least upper bound.

So there exists x_n such that $f(x_n) > M - \frac{1}{n}$. Now (x_n) is a sequence in X. X is compact, and hence sequentially compact, and there exists a subsequence (x_{n_k}) such that

$$\lim_{k\to\infty} x_{n_k} = x_0$$

Now, since f is continuous

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$$

But

$$M > f(x_{n_k}) > M - \frac{1}{n}$$

As $k \to \infty$, $n_k \to \infty$, so $\lim_{k\to\infty} f(x_{n_k}) = M = f(x_0)$¿ So f achieves its maximum value at x_0. Hence, the statement follows. ☐

Example 20.7 $f(x) = \frac{1}{x}$ on $(0, \infty)$. f is continuous, $M = \sup(f(X)) = \infty$ and $m = \inf(f(X)) = 0$, but f never achieves those values. But that's because $(0, \infty)$ is not compact.

§20.3 Lebesque Number

X- metric space, $\{O_\alpha\}_{\alpha\in I}$ an open cover of X,

$$X = \bigcup_{\alpha\in} O_\alpha$$

Take $x \in X$. Then $x \in O_\alpha$ and so there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset O_\alpha$. But ε depends on X. However, if X is compact, one can find ε_0 such that $B(x, \varepsilon_0) \subset O_\alpha$ for some α and all x.

Theorem 20.8 ***(Lebesque Covering Lemma)*** *Let X be a compact metric space. Let $\{O_\alpha\}_{\alpha\in I}$ be an open cover of X. Then there exists $\varepsilon > 0$ such that for all $x \in X$, $B(x, \varepsilon)$ is contained in an element of the cover. Such an ε is called a Lebesque number of the cover.*

Proof. We will argue by contradiction. Suppose that the statement is not true, namely that is not true that

$$(\exists\varepsilon > 0)(\forall x \in X)(\exists\alpha \in I)(B(x, \varepsilon) \subset O_\alpha)$$

i.e.

$$(\forall\varepsilon > 0)(\exists x \in X)(\forall\alpha \in I)(B(x, \varepsilon) \not\subset O_\alpha)$$

Take $\varepsilon = \frac{1}{n}$. Then $\exists x_n$ such that $B\left(x_n, \frac{1}{n}\right)$ is not contained in any single member of the cover.

The sequence (x_n) has a convergent subsequence (X is sequentially compact).

Let (x_{n_k}) be a converging subsequence

$$\lim_{k\to\infty} x_{n_k} = x_0$$

$x_0 \in O_{\alpha_0}$ and since O_{α_0} is open, there is $r > 0$ such that $B(x_0, r) \subset O_{\alpha_0}$.

Now, let n_k be such that

$$d(x_0, x_{n_k}) < \frac{r}{2} \text{ and } \frac{1}{n_k} < \frac{r}{2}$$

$$\left(\lim_{k\to\infty} d(x_0, x_{n_k}) = 0 \text{ and } n_k \to \infty \text{ as } x \to \infty\right)$$

Look at $B\left(x_{n_k}, \frac{1}{n_k}\right)$. If $y \in B\left(x_{n_k}, \frac{1}{n_k}\right)$, then

$$\begin{aligned} d(x_0, y) &\le d(x_0, x_{n_k}) + d(x_{n_k}, y) \\ &< \frac{r}{2} + \frac{1}{n_k} \\ &< \frac{r}{2} + \frac{r}{2} = r \end{aligned}$$

So,

$$B\left(x_{n_k}, \frac{1}{n_k}\right) \subset B(x_0, r) \subset O_{\alpha_0}$$

So, $B\left(x_{n_k}, \frac{1}{n_k}\right)$ is contained in a single element of the cover. Contradiction. $\square$

Theorem 20.9 *Let (X, d_x) and (Y, d_y) be metric spaces and $f : X \to Y$ be a continuous function. Suppose X is compact. Then f is uniformly continuous.*

§21 Thursday, October 10

Theorem 21.1 *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a continuous function. Suppose that X is compact. Then the function f is uniformly continuous.*

Proof. We want to prove

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$$

We know that f is continuous and that X is compact. Fix $\varepsilon > 0$. Since f is continuous, for any $x \in X$ there is a $\delta_x > 0$ such that

$$d_X(x, y) < \delta_x \implies d_Y(f(x), f(y)) < \frac{\varepsilon}{2}$$

Let $O_x = B(x, \delta_x)$ and $\{O_x\}_{x \in X}$ an open cover of X.

$$X = \bigcup_{x \in X} O_x$$

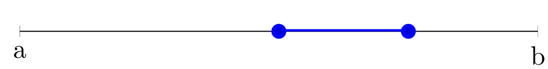
If $U, v \in O_x$, then

$$d_Y(f(u), f(v)) \le d_Y(f(u), f(x)) + d_Y(f(x), f(u)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Because X is compact, there exists a Lebesque number $\delta > 0$ for the cover $\{O_x\}_{x \in X}$. This δ (give our ε) suffices for uniform continuity. To see this, let $x, y \in X$ such that $d_X(x, y) < \delta$. THen $y \in B(x, \delta)$. Since δ is the Lebesque number, there exists $x' \in X$ such that $B(x, \delta) \subset O_{x'}$. But then $x, y \in O_{x'} \implies d_Y(f(x), f(y)) < \varepsilon$. $\square$

Example 21.2 (Basic fact from Analysis 2). A closed interval $[a, b]$ is a compact subset of $\mathbb{R}$. In other words, any sequence $(x_n)_{n=1}^{\infty}$ in $[a, b]$ has a convergent subsequence.

Proof:



Divide $[a, b]$ in two parts of equal length. One of them must contain infinitely many elements of the sequence; call this interval I_1, and denote the first element as x_{n_1}. Then divide I_1 into two equal parts. One of those parts has to contain infinitely many elements of the sequence $\{x_n : n > n_1\}$. Pick one such part, denote it by I_2, and let x_{n_2} be the first such element in I_2. Continue inductively. In this way, we have constructed a sequence of intervals I_k such that the length of I_k is $\frac{b-a}{2^k}$ and a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_j} \in I_k$ for all $j \geq k$. Sequence x_{n_k} is Cauchy: Since $x_{n_j}, x_{n_{j'}} \in I_k$ for $j, j' \geq k$,

$$|x_{n_j} - x_{n_{j'}}| \leq \frac{b-a}{2^k}$$

Given $\varepsilon > 0$, $\frac{b-a}{2^k} < \varepsilon$ for k large enough.

The completeness of $\mathbb{R}$ implies that $\lim_{k\to\infty} x_{n_k} = x$ exists. Since $a \leq x_{n_k} \leq b$, $a \leq x \leq b$.

Example 21.3 Let X be a set and d the discrete metric on X,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then X is compact if and only if X is a finite set.

Any metric space consisting of finitely many points is compact (the trivial open cover is its own finite subcover). Suppose that (X, d) is compact.

$$B(x, 1/2) = \{x\}$$

So $X = \bigcup_{x\in X}\{x\}$ is an open cover. Since we assumed (X, d) is compact, we can pick a finite subcover: $\exists x_1, \ldots, x_n \in X$ such that

$$X = \{x_1\} \cup \ldots \cup \{x_n\} = \{x_1, \ldots, x_n\}$$

So X consists of finitely many points.

§21.1 Compactness in finite dimensional normed spaces

§21.1.1 Preliminaries

Let X be a finite dimensional vector space. Let $\{e_1, \ldots, e_N\}$ be a basis of X, with $\dim X = N$. So any $x \in X$ can be written as

$$x = \alpha_1 e_1 + \ldots + \alpha_N e_N$$

Let $\|x\|_\infty = \max_{1\leq k\leq n} |\alpha_k|$.

Exercise 21.4 $\|\cdot\|_\infty$ is a norm on$X$.

Note that $\|\cdot\|_\infty$ depends on the choice of basis.

Proposition 21.5 Let $S \subset X$. T.F.A.E.

1. S is closed and bounded in $(X, \|\cdot\|_\infty)$.

2. S is compact in $(X, \|\cdot\|_\infty)$.

Proof. We need to prove (1) $\implies$ (2) ((2) $\implies$ (1) is true in any metric space).

We will prove that S is sequentially compact. Let (x_n) be a sequence in S. Since S is bounded, $\exists M > 0$ such that $\|x_n\|_\infty \leq M$. for all n. Now,

$$x_n = \alpha_1^{(n)} e_1 + \ldots + \alpha_N^{(n)} e_N$$

And

$$\|x_n\|_\infty = \max_{1\leq j\leq N} |\alpha_j^{(n)}|$$

So for all j, $(\alpha_j^{(n)})_{n=1}^{\infty}$ is a bounded sequence of real numbers. Such a sequence always has a convergent subsequence.

So $\alpha_1^{(n)}$ has a convergent subsequence with indicies, say n_{k_1}. Then $\alpha_2^{(n_{n_1})}$ has a convergent subsequence of this subsequence; call those indicies n_{k_2}. We can repeat this for the finitely many α_j to find a subsequence for which $\alpha_N^{(n_{k_N})}$ converges. But this a subsequence of a sequence for which each α_j for $j < N$ converges. Call the indicies of this last subsequence α_{n_k}.

Then

$$\lim_{k\to\infty} \alpha_j^{(n_k)} = \alpha_j$$

for some α_j.

□

§22 Friday, October 11, 2013

§22.1 Compactness in finite-dimensional normed spaces (ctd.)

Last class: X-vector space, $\dim X = N < \infty$, $\{e_1, \ldots, e_N\}$ given basis.

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n$$

$$\|x\|_\infty = \max_{1\le k\le N} |\alpha_k|$$

is a norm on X.

$S \subset X$ is compact on $(X, \|\cdot\|_\infty)$, $|f|$ is closed and bounded.

Theorem 22.1 *Let X be finite dimensionsal vector space and $\|\cdot\|_1$, $\|\cdot\|_2$ two norms on X. Then these norms are equivalent, that is to say, there exists constants $C_1, C_2 > 0$ such that*

$$C_1\|x\|_2 \le \|x\|_1 \le C_2\|x\|_2$$

for all $x \in X$.

Proof. Fix a basis $\{e_1, \ldots, e_N\}$ of X. And let $\|\cdot\|_\infty$ be as in the previous proposition.

Let $\|\cdot\|$ be any other norm on X. We will show that $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent.

Let $x \in X$. Expand,

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n$$

Then,

$$\|x\| = \|\sum_{j=1}^N \alpha_j e_j\| \le \sum_{j=1}^N \|\alpha_j e_j\| = \sum_{j=1}^N |\alpha_j|\|e_j\| \le \left(\max_{1\le j\le N} |\alpha_j\right) \sum_{j=1}^N \|e_j\|$$

$$C_2 = \sum_{j=1}^N \|e_j\|$$

is a finite number that only depends on $\|\cdot\|$ and a choice of basis.

So, $\|x\| \le C_2 \cdot \|x\|_\infty$.

Now we need the lower bound. Define a function $f : X \to \mathbb{R}$ by $f(x) = \|x\|$. Then

$$|f(x) - f(y)| = |\|x\| - \|y\|| \le \|x - y\| \le C_2\|x - y\|_\infty$$

So f is Lipschitz as a function on normed space $(X, \|\cdot\|_\infty)$. In particular, f is continuous. Let now

$$S = \{x \in X : \|x\|_\infty = 1\}$$

S is the unit sphere in $(X, \|\cdot\|_\infty)$.

S is obviously bounded in $(X, \|\cdot\|_\infty)$ and S is closed in $(X, \|\cdot\|_\infty)$: if $x_n \in S$ and $x_n \to x$ as $n \to \infty$, then $\Big|\|x_n\|_\infty - \|x\|_\infty\Big| \le \|x_n - x\|_\infty$. So $\Big|1 - \|x\|_\infty\Big| \le \|x_n - x\|_\infty \to 0$ as $n \to \infty$. So $\|x\|_\infty = 1$.

By previous proposition, S is compact in $(X, \|\cdot\|_\infty)$. The function $f(x) = \|x\|$ is continuous on $(X, \|\cdot\|_\infty)$ and so is continuous when restricted to any metric subspace. Since S is compact, f achieves its minimum values. So there exists $x_0 \in S$ such that

$$m = \|x_0\| = \inf_{x \in S} \|x\|$$

We must have that $m > 0$. Otherwise, if $m - 0$, we have $x_0 = 0_X$ but $x_0 \in S$, $\|x_0\|_\infty = 1$, contradiction.

So $m > 0$. Let now $x \in X$ arbitrary subject to $x \neq 0$. Then $\frac{x}{\|x\|_\infty} \in S$.

$$\left\| \frac{x}{\|x\|_\infty} \right\| = 1$$

So,

$$\left\| \frac{x}{\|x\|_\infty} \right\| \geq m \iff \frac{1}{\|x\|_\infty} \cdot \|x\| \geq m \text{ or } \|x\| \geq m\|x\|_\infty$$

for all $x \in X$ with $x \neq 0$. So,

$$\|x\| \geq m \cdot \|x\|_\infty$$

which is obviously true when $x = 0$. So, $\|x\| \geq C_1 \|x\|_\infty$ where $C_1 = m > 0$. So, $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent. Since equivalence of a norm is an equivalence relation, any two norms are equivalent. $\square$

Theorem 22.2 ***(Heine-Borel)*** *Let $(X, \|\cdot\|)$ be finite dimensional normed vector space and let $S \subset X$. Then T.F.A.E.*

1. *S is closed and bounded*
2. *S is compact in $(X, \|\cdot\|)$*

Proof. (2) $\implies$ (1): general fact

(1) $\implies$ (2): Fix a basis $\{e_1, \ldots, e_N\}$ and let $\|\cdot\|_\infty$ be the corresponding norm. Since $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent, there exist constants C_1 and C_2 such that

$$\forall x \in X, C_1\|x\| \leq \|x\|_\infty \leq C_2\|x\| \tag{$\star$}$$

S is closed and bounded in $(X, \|\cdot\|)$. THen the first inequality in $(\star)$ gives that S is bounded in $(X, \|\cdot\|_\infty)$ and the second inequality gives that it is also closed in $(X, \|\cdot\|_\infty)$. Why? Let $x \in \text{cl}(S)$ in $(X, \|\cdot\|_\infty)$. Then there is $x_n \in S$ such that $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$. Then $\|x - x_n\| \leq \frac{1}{C_2}\|x - x_n\|_\infty$. So, $x_n \to x$ in $(X, \|\cdot\|)$. Since S is closed in $(X, \|\cdot\|)$, $x \in S$.

So, S is compact in $(X, \|\cdot\|_\infty)$ by previous proposition. Let (x_n) be a sequence in S. Since S compact in $(X, \|\cdot\|_\infty)$, this sequence has a converging subsequence x_{n_k} converging to an element $x \in S$ in $(X, \|\cdot\|_\infty)$. Since the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent, $x_{n_k} \to x$ in $(X, \|\cdot\|)$ also. Hence, S is sequentially compact, and so is compact.

$\square$

§22.2 Compactness in infinite dimensional normed spaces

"Let $(X, \|\cdot\|)$ be a normed space in which the unit ball is compact"

Theorem 22.3 *Let $(X, \|\cdot\|)$ be a normed space. Suppose that $\dim X = \infty$. Then the closed unit ball*

$$B = \{x \in X : \|x\| \leq 1\}$$

is not a compact subset of X.

We need two lemmata.

Lemma 22.4 (Lemma 1, Riesz Lemma) Let $(X, \|\dot{c}\|)$ be a normed space and $S \subset X$ a proper closed subspace of X. Let $0 < \alpha < 1$. Then there exists $y_\alpha \in X \setminus S$ such that $\|y_\alpha\| = 1$ and

$$\|y_\alpha - x\| > \alpha \quad \forall x \in S$$

Proof. Since S is a proper subspace, $\exists y \in X \setminus S$. Let

$$d = \operatorname{dist}(y, S) = \inf\{\|y - x\| : x \in S\}$$

Claim: $d > 0$.

If $d = 0$, then for any $n > 0$, $\frac{1}{n}$ is not a lower bound. For our set $\exists x_n \in S$ such that $\|y - x_n\| < \frac{1}{n}$. Then $x_n \to y$ and since S is closed, $y \in S$ (contradiction).

Now, $\frac{d}{\alpha} > d$ since $0 < \alpha < 1$. So $\frac{d}{\alpha}$ is not a lower bound for the set $\{\|y - x\| : x \in S\}$. So there is an $x_\alpha \in S$ such that

$$\|y - x_\alpha\| < \frac{d}{\alpha} \iff \frac{1}{\|y - x_\alpha\|} > \frac{\alpha}{d}$$

Let $y_\alpha = \frac{y - x_\alpha}{\|y - x_\alpha\|}$. Then $\|y_\alpha\| = 1$.

Also, $y_\alpha \notin S$. If $y_\alpha \in S$, then $\|y - x_\alpha\| y_\alpha + x_\alpha = y$. So y is a linear combination of elements of S, and so is in S too (contradiction).

For $x \in X$,

$$\|y_\alpha - x\| = \left\| \frac{y - x_\alpha}{\|y - x_\alpha\|} - x \right\| = \frac{1}{\|y - x_\alpha\|} \left\| y - \underbrace{(x_\alpha + \|y - x_\alpha\| x)}_{\in S} \right\| \geq \frac{1}{\|y - x_\alpha\|} \cdot d \geq \frac{\alpha}{d} \cdot d = \alpha$$

□

§23 Tuesday, October 15, 2013

Last class:

Theorem 23.1 *Let $(X, \|\cdot\|)$ be a normed space. Suppose that $\dim X = \infty$. Then the closed unit ball*

$$B = \{x \in X : \|x\| \leq 1\}$$

is not *a compact subset of X.*

Lemma 23.2 (Lemma 1, Riesz) Let $(X, \|\cdot\|)$ be a normed space. Let $S \subset X$ be a proper closed subspace of X and $0 < \alpha < 1$. Then there exists $y_\alpha \in X \setminus S$, $\|y_\alpha\| = 1$, such that

$$\|y_\alpha - x\| > \alpha$$

for all $x \in S$.

Lemma 23.3 (Lemma 2) Let $(X, \|\cdot\|)$ be a normed space and $S \subset X$ a finite dimensional subspace. Then S is closed.

Proof. Let $x \in \operatorname{cl} S$. Then there exists sequence $(x_n)_{n=1}^\infty$ in S such that

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

Now (x_n) is converging, hence (x_n) is bounded. That is, $\exists M > 0$ such that

$$\|x_n\| \leq M$$

Consider $(S, \|\cdot\|)$, finite dimensional normed subspace in X. The sequence x_n is contained in the closed ball $\{x \in S : \|x\| \leq M\}$, which is a closed bounded subset of S. Since S is finite dimensional, this closed ball is compact. Hence, there exists a subsequence of x_{n_k} that converges to an element $y \in S$.

$$\lim_{k\to\infty} \|x_{n_k} - y\| = 0$$

Also,

$$\lim_{k\to\infty} \|x_{n_k} - x\| = 0$$

So $x = y$ and $x \in S$. So $\operatorname{cl} S \subset S$ and S is closed. □

§23.1 Proof of the Theorem

Take $x_1 \in B$, $\|x_1\| = 1$. Let $S_1 = \operatorname{span}\{x_1\} = \{\alpha x_1 : \alpha \in \mathbb{R}\}$. This is a closed vector subspace of X and $S_1 \neq X$ since X is not finite dimensional. So, there exists $x_2 \in X$, $x_2 \notin S_1$, $\|x_2\| = 1$, and

$$\|x_2 - x_1\| \geq \frac{1}{2}$$

By the Riesz lemma.

Let now $S_2 = \operatorname{span}\{x_1, x_2\} = \{\alpha_1 x_1 + \alpha_2 x_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$.

S_2 is a closed proper subspace of X and we can find x_3 such that $\|x_3\| = 1$ and $\|x_3 - x_2\| \geq \frac{1}{2}$ and $\|x_3 - x_1\| \geq \frac{1}{2}$.

So, inductively, we construct a sequence $(x_n)_{n=1}^{\infty}$ such that $\|x_n\| = 1$ and $\|x_n - x_m\| \geq \frac{1}{2}$ where $n \geq m$.

This sequence cannot have a convergent subsequence because all subsequences are not Cauchy. So B is not sequentially compact, and so not compact.

§23.2 Products of Compact Metric Spaces

Exercise 23.4 $(x_1, d_1), \ldots, (X_n, d_n)$ finitely many compact metric spaces, (X, d) their product. Then X is compact.

We will prove a more general fact:

Proposition 23.5 Let (X_k, d_k) $k = 1, 2, \ldots$ be a sequence of compact metric spaces and (X, d) their product:

$$X = \{x = (x_k)_{k=1}^{\infty}, x_k \in X_k\}$$

and

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

Then (X, d) is compact.

Remark. This is a special case of Tychonoff theorem.

Proof. The proof is based on Cantor's diagonalization argument. We want to show that a given sequence $x^{(n)}$ in (X, d) has a convergent subsequence. Convergence in the product space is equivalent to componentwise convergence. So we need to show there exists a subsequence $x^{(n_j)} = (x_k^{(n_j)})_{k=1}^{\infty}$, $j = 1, 2, \ldots$ such that for every k, the limit

$$\lim_{j \to \infty} x_k^{(n_j)} \text{ exists}$$

$x_1^{(n)}$, $n = 1, 2, \ldots$, is a sequence of the first components. Since X_1 is compact, we can extract a convergent subsequence.

$$x_1^{(n_{11})}, x_1^{(n_{21})}, \ldots, x_1^{(n_{j1})}, \ldots$$

Now, look at $X_2^{(n)}$ along the same sequence of indicies:

$$x_2^{(n_{11})}, x_2^{(n_{21})}, \ldots, x_2^{(n_{j1})}, \ldots$$

and extract a convergent subsequence:

$$x_2^{(n_{12})}, x_2^{(n_{22})}, \ldots, x_2^{(n_{j2})}, \ldots$$

and we continue. Suppose we get

$$x_{j-1}^{(n_{1,j-1})}, x_{j-1}^{(n_{2,j-1})}, x_{j-1}^{(n_j,j-1}, \ldots$$

Then we look at $x_j^{(n_{1,j-1})}, x_j^{(n_{j,j-1})}, \ldots$ and we pick out a convergent subsequence:

$$x_j^{(n_{1,j})}, x_j^{(n_{2,j}}), \ldots, x_j^{(n_{j,j})}, \ldots$$

This procedure does not stop (we have an infinite product). We don't have a last one, but we can pick up a diagonal subsequence:

$$x^{(n_{jj})} = (x_k^{(n_{jj})})_{n=1}^{\infty}$$

Now, for every k,

$$\lim_{j\to\infty} x_k^{(n_{jj})} \text{ exists}$$

and we are done.

Why does this work?

$$\begin{array}{llll}
n_{11} & n_{21} & n_{31} & \cdots \\
n_{12} & n_{22} & n_{32} & \cdots \\
\vdots & & & \\
n_{1,j-1} & n_{2,j-1} & \cdots & \\
n_{1,j} & n_{2,j} & \cdots n_{jj} & \cdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

So $x_1^{(n_{jj})}$, $j = 1, 2, \ldots$ is a subsequence of $x_1^{(n_{j1})}$ so it converges.

Likewise, $x_2^{(n_{jj})}$, $j = 1, 2, \ldots$ is a subsequence of $x_2^{(n_{j2})}$ except possibly for the first element, so it converges.

$x_k^{(n_{jj})}$ is a subsequence of a converging sequence $x_k^{(n_{jk})}$ in X_k except possibly for the first $k-1$ elements (which does not affect the convergence).

□

§23.3 9.6 Separable metric spaces

Countable and uncountable sets (Section 1.3 in the book; "READ it").

Definition 23.6 A set S is called countable if there exists a surjection (onto map) $f : \mathbb{N} \to S$. That means the set S can be enumerated as $S = \{s_1, s_2, \ldots\}$, where $f(k) = s_k$. S could be a finite set. If S is countable and infinite, then f is a bijection between $\mathbb{N}$ and S.

If a set is not countable, it is called uncountable.

Proposition 23.7

1. If S is countable and $S' \subset S$, then S' is countable.
2. If $S_1, \ldots, S_n$ are countable sets, their cartesean product $S = S_1 \times \ldots \times S_n = \{(s = (s_1, \ldots, s_n) : s_k \in S_k\}$ is countable
3. A countable union of countable sets is countable. i.e. if $\{S_i\}_{i\in I}$ is a family of countable sets and the index I is a countable set, then $\bigcup_{i\in I} S_i$ is countable.

Remark. A countable product of countable sets is *not* countable (in general).

Example 23.8 Set of rationals $\mathbb{Q}$ is countable.

Proof: $\mathbb{Z}$ is countable: $f : \mathbb{N} \to \mathbb{Z}$ such that $f(2n) = n$ and $f(2n-1) = -(n-1)$. So $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable. Then the map $(m, n) \to \frac{m}{n}$ is a surjection between $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and $\mathbb{Q}$.

Example 23.9 $a < b$, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is not countable.

§24 Thursday, October 17, 2013

§24.1 Separable Metric Spaces

Countable/uncountable sets. S is countable if an onto map $f : \mathbb{N} \to S$ exists.

Proposition 24.1 Let S be the set of all sequences of 0's and 1's. $S = \{(s_n)_{n=1}^{\infty} : s_n \in \{0,1\}\}$. Then S it not countable.

Proof. Suppose that S is countable. Since S is obviously infinite, there exists a bijection $f : \mathbb{N} \to S$ and we can enumerate $S = \{s^{(1)}, s^{(2)}, \ldots\}$. Let now $s = (s_n)$ be a sequence of 0's and 1's constructed as follows:

$$s_n = \begin{cases} 1 & \text{if } s_n^{(n)} = 0 \\ \\ 0 & \text{if } s_n^{(n)} = 1 \end{cases}$$

(s_n) is a sequence of 0's and 1's, so $s \in S$. But $s = (s_n)$ differs from any $s^{(n)}$ since their n^{th} element differs. Then $s = (s_n) \notin S$, which is a contradiction. □

Exercise 24.2 Using previous proposition, prove that the interval $[0,1]$ in $\mathbb{R}$ is not countable.

Hint: $S, s = (s_n), s_n \in \{0,1\}$, $f : S \to [0,1]$, $f(s) = \sum_{n=1}^{\infty} s_n 2^{-n}$. From the binary representation of $x \in [0,1]$, f is onto. BUt f is not one-to-one.

$$(1,0,0,\ldots,0,\ldots) \overset{f}{\mapsto} \frac{1}{2}$$

$$(0,1,1,\ldots,1,\ldots) \overset{f}{\mapsto} \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}$$

Let S_0 be the set of all sequences in S that terminate in 1's. Then S_0 is countable. $S_1 = S \setminus S_0$, S is not countable. The map

$$f : S_1 \mapsto [0,1]$$

$$f(s) = \sum_{n=1}^{\infty} s_n 2^{-n}$$

is 1 to 1 and onto. So $[0,1]$ is in bijection with S_1, which is uncountable.

Write down the details.

Definition 14.8 A subset S of a metric space (X,d) is called dense if

$$\mathrm{cl}(S) = X$$

In other words, S is dense in X if for any $x \in X$, there is a sequence (x_n) in S such that $\lim_{n\to\infty} x_n = x$.

Proposition 24.3 (Compare with the book) T.F.A.E.

1. S is dense in X
2. Any non-empty open set in X contains an element of S.

Proof. (1) $\implies$ (2): Suppose (2) fails, that is, that there exists a non-empty open set O such that $O \cap S = \varnothing$. The $S \subset O^c$ and O^c is closed. Then $\mathrm{cl}(S) \subset O^c \neq X$, so X is not dense. This is a contradiction.

(2) $\implies$ (1): Suppose (1) fails. Then $\mathrm{cl}(S) \neq X$. Let $O = [\mathrm{cl}\, S]^c$. Then O is non-empty, open, and $O \cap \mathrm{cl}(S) = \varnothing$, contradicting (2). □

Definition 24.4 A metric space X is called separable if it contains a countable dense set.

Example 24.5 $\mathbb{R}$ is separable. The rationals $\mathbb{Q}$ are a countable dense set.

Example 24.6 Let $(X, \|\cdot\|)$ be a normed space, $\dim X < \infty$. Then X is separable.

Proof: Let $\{e_1, \ldots, e_N\}$ be a basis of X. Let $S = \{r_1 e_1 + \ldots r_N e_N : r_k \in \mathbb{Q}, 1 \le k \le N\}$. Then the map

$$f : S \to \underbrace{\mathbb{Q} \times \ldots \times \mathbb{Q}}\, N \text{ times} \qquad f(x) = (r_1, \ldots, r_N)$$

if $x = r_1 e_1 + \ldots + r_N e_N$.

f is a bijection, $\underbrace{\mathbb{Q} \times \ldots \times \mathbb{Q}}$ N times is countable, so S is countable.

Let $x \in X$ and $\varepsilon > 0$. Write

$$x = \alpha_1 e_1 + \ldots + \alpha_N e_N$$

Pick a rational number r_k such that $|\alpha_k - r_k| < \frac{\varepsilon}{N\|e_k\|}$ for each $1 \leq k \leq N$.

Let $y = \sum_{k=1}^{N} r_k e_k \in S$. Then

$$\|x - y\| = \left\| \sum_{k=1}^{N} (\alpha_k - r_k) e_k \right\| \leq \sum_{k=1}^{N} |\alpha_k - r_k| \cdot \|e_k\| < \sum_{k=1}^{N} \frac{\varepsilon}{N\|e_k\|} \|e_k\| = \varepsilon$$

Important: If $(X, \|\cdot\|)$ is infinite dimensional, it could be separable, but could also be non-separable.

Proposition 24.7 Any compact metric space is separable.

Proof. Let X be compact. Then X is totally bounded: for any $\varepsilon > 0$, X can be covered by finitely many open balls of radius ε.

Let $N > 0$ be an integer, $\varepsilon = \frac{1}{N}$, and let $x_1^{(N)}, \ldots, x_{n_N}^{(N)}$ be points in X such that

$$X = B\left(x_1^{(n)}, \frac{1}{N}\right) \cup \ldots \cup B\left(x_{n_N}^{(n)}, \frac{1}{N}\right)$$

Let $S = \bigcup_{N=1}^{\infty} \{x_1^{(N)}, \ldots, x_{n_N}^{(N)}\}$. Then S is a countable union of countable sets, so is countable.

Let now $x \in X$. Then $\exists k_N$, $1 \leq k_N \leq n_N$, such that $x \in B\left(x_{k_N}^{(N)}, \frac{1}{N}\right)$ for each $N = 1, 2, \ldots$.

Since $d(x, x_{k_N}^{(N)}) < \frac{1}{N}$, $(x_{k_N}^{(N)})_{N=1}^{\infty}$ is a sequence in S that converges to x. Hence, $\operatorname{cl} S = X$. ☐

Proposition 24.8 Let X be a metric space. T.F.A.E.:

1. X is separable
2. There exists a countable collection $\{O_n\}_{n=1}^{\infty}$ of non-empty open sets in X such that every open set in X can be written as the union of a subcollection of $\{O_n\}_{n=1}^{\infty}$

Proof. (1) $\implies$ (2): Let $S = \{x_n\}_{n=1}^{\infty}$ be a countable dense set in X. Consider a family of open sets:

$$\{B\left(x_n, \frac{1}{m}\right) : n, m \in \mathbb{N}\} \tag{$\star$}$$

This is a countable collection since it is indexed by a countable set $(n, m) \in \mathbb{N} \times \mathbb{N}$.

Claim is: any open set in X can be written as a union of a subcollection of $(\star)$. Let O be an open set. Let $x \in O$. Then $\exists m > 0$ such that

$$B\left(x, \frac{1}{m}\right) \subset O$$

Note: m depends on $x; m = m_x$.

Now, since S is dense, we can find n such that

$$d(x_n, x) < \frac{1}{2m_x}$$

This n also depends on x; $n = n_x$.

Obviously, $x \in B\left(x_{n_x}, \frac{1}{2m_x}\right)$. If $z \in B\left(x_{n_x}, \frac{1}{2m_x}\right)$,

$$d(x, z) \leq d(x, x_{n_x}) + d(x_{n_x}, z) < \frac{1}{2m_x} + \frac{1}{2m_x} = \frac{1}{m_x}$$

So $z \in B\left(x, \frac{1}{m_x}\right)$. So we have shown that for any $x \in O$, there are integers $n_x, m_x \in \mathbb{N}$ such that

$$x \in B\left(x_{n_x}, \frac{1}{2m_x}\right) \subset B\left(x, \frac{1}{m_x}\right) \subset O$$

So we will take a union:

$$O = \bigcup_{x \in O} B\left(x_{n_x}, \frac{1}{2m_x}\right)$$

To be pedagogical, $I = \{(n_x, 2m_x) : x \in O\}$. Then

$$O = \bigcup_{(n,m) \in I} B\left(x_n, \frac{1}{m}\right)$$

Now, the converse: (2) $\implies$ (1): We know that there exists $\{O_n\}_{n=1}^{\infty}$ such that any open set in X is union of $\{O_n\}_{n=1}^{\infty}$ where the O_n are non-empty and open. Pick $x_n \in O_n$ and let $S = \{x_n\}_{n=1}^{\infty}$. Then any open set O in X contains some O_n and hence contains x_n. So any non-empty open set in X has non-empty intersection with S. By the first proposition of today, S is dense. □

§25 Friday, October 18, 2013

Reminder: Friday $\iff$ tutorial

Proposition 24.8 Let X be a metric space. T.F.A.E.:

1. X is separable
2. There exists a countable collection $\{O_n\}_{n=1}^{\infty}$ of non-empty open sets in X such that every open set in X can be written as the union of a subcollection of $\{O_n\}_{n=1}^{\infty}$

Proposition 25.1 Let X be a metric space and $Y \subset X$ a metric subspace of X. If X is separable, then Y is also separable.

Proof. We will use the last proposition.

We want to prove: there exist countable collection $\{V_n\}_{n=1}^{\infty}$ of non-empty open sets in Y such that any open set in Y can be written as a union of some subcollection of $\{V_n\}_{n=1}^{\infty}$.

What we know: There exist family $\{O_n\}_{n=1}^{\infty}$ of non-empty open sets in X such that every open set in X can be written as a union of some subcollection of $\{O_n\}$.

Set

$$V_n = Y \cap O_n$$

V_n is open in Y. So we have a family $\{V_n\}_{n=1}^{\infty}$. If some V_n are empty, just drop them and re-enumerate the sequence. Let now $V \subset Y$ be open. Then we know there exists an open set $O \subset X$ such that

$$V = Y \cap O$$

Now, there exist $I \subset \mathbb{N}$ such that

$$O = \bigcup_{i \in I} O_i$$

But then

$$V = Y \cap O = \bigcup_{i \in I} (Y \cap O_i) = \bigcup_{i \in I} V_i$$

Hence, $\{V_n\}_{n=1}$ is a suitable countable collection. □

Proposition 25.2 Let X be a separable metric space and $\{B(x_i, r_i)\}_{i \in I}$ a family of disjoint open balls in X. Then I has to be countable.

Exercise 25.3 Prove this using proposition 24.8.

Proof. Let $\{y_1, y_2, \dots\}$ be a countable dense set in X. Then for every $i \in I$, there exists $n(i) \in \mathbb{N}$ such that

$$y_{n(i)} \in B(x_i, r_i)$$

Now, if $i \neq i'$, then, since balls are disjoint, $y_{n(i)} \neq y_{n(i')}$, and so $n(i) \neq n(i')$.

Consider the map

$$t : I \to \mathbb{N} \qquad f(i) = n(i)$$

f is one-one so I is in bijective correspodence with subset of $\mathbb{N}$. So I is countable. □

Remark. The last proposition is very useful in showing lack of separability. Given metric space X, if we can find an uncountable collection of disjoint open balls in X, then X is *not* separable.

Problem 4, homework 4: X, functions of finite variations $f : [a, b] \to \mathbb{R}$, $\|f| = |f(a)| + V_a^b f$. X is not separable.

Sketch of the solution: $a < c < b$

$$f_c(x) = \begin{cases} 0 & \text{if } a \leq x < c \\ 1 & \text{if } c \leq x \leq b \end{cases}$$

Show that for $c \neq c'$, $\|f_c - f_{c'}\| \geq 2$.

Consider $\left\{B\left(f_c, \frac{1}{2}\right)\right\}_{c \in (a,b)}$. This is an uncountable family of disjoint balls, so X has to be non-separable.

§25.1 Density of polynomials

$C([0, 1])$ is the vector space of all continuous functions

$$f : [0, 1] \to \mathbb{R}$$

with the norm $\|f\|_\infty = \max_{a \leq x \leq n} |f(x)|$. This is a Banach space.

Polynomial:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$

Theorem 25.4 *Let $f \in C([0, 1])$ and $\varepsilon > 0$. Then there exists a polynomial $p(x)$ such that*

$$\|f - p\|_\infty = \max_{a \leq x \leq b} |f(x) - p(x)| < \varepsilon$$

Moreover, p can be chosen in the following concrete way:

Let

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}$$

Then for any $\varepsilon > 0$, there exists $n(\varepsilon)$ such that for $n \geq n(\varepsilon)$,

$$\|f - p_n\| < \varepsilon$$

Remark. Polynomials are called Bernstein polynomials.

Remark. If f is Lipschitz, then one can estimate $n(\varepsilon)$ in terms of ε. This is important for numerics.

Remark. The idea comes from probability.

Proof.
We will start with some identities.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$x\frac{\partial}{\partial x}(x+y)^n = \sum_{k=0}^{n} k\binom{n}{k} x^k y^{n-k}$$

So

$$x \cdot n(x+y)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^k y^{n-k}$$

Take $y = (1-x)$. Then

$$\boxed{nx = \sum_{k=0}^{n} k\binom{n}{k} x^k (1-x)^{n-k}} \tag{a}$$

We can take another derivative:

$$x^2\frac{\partial}{\partial x^2}(x+y)^n = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k y^{n-k}$$

So

$$x^2 n(n-1)(x+y)^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k y^{n-k}$$

Take $y = 1-x$. Then

$$x^2 n(n-1) = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} \tag{b}$$

We can add equation (a) to (b) to get

$$\boxed{n(n-1)x^2 + nx = \sum_{k=0}^{n} k^2\binom{n}{k} x^k (1-x)^{n-k}} \tag{c}$$

Shorthand: $r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$.
Notice we have these properties:

$$\sum_{k=0}^{n} r_k(x) = 1 \tag{1}$$

$$\sum_{k=0}^{n} k r_k(x) = nx \tag{2}$$

$$\sum_{k=0}^{n} k^2 r_k(x) = n(n-1)x^2 + nx \tag{3}$$

One more formula:

$$\begin{aligned}\sum_{k=0}^{n}(nx-k)^2 r_k(x) = \sum_{k=0}^{n} n^2x^2 r_k(x) - \sum_{k=0}^{n} 2knx r_k(x) + \sum_{k=0}^{n} k^2 r_k(x) &= n^2x^2 - 2n^2x^2 + n(n-1)x^2 + nx \\ &= -nx^2 + nx\end{aligned}$$

$$\sum_{k=0}^{n}(nx-k)^2 r_k(x) = nx(1-x) \tag{4}$$

Let $f \in C([0,1])$ and $\varepsilon > 0$.

f is continuous and hence uniformly continuous on $[0,1]$. Then there exist $\delta > 0$ such that $\forall x, y \in [a,b]$, $|x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$.

Let $M = \|f\|_\infty = \max_{a \le x \le b} |f(x)|$.

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x)$$

where $r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Then

$$\begin{aligned}
|f(x) - p_n(x)| &= \left| f(x) \sum_{k=0}^{n} r_k(x) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x) \right| \\
&= \left| \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right) \right) r_k(x) \right| \\
&= \left| \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| < \delta}} \left(f(x) - f\left(\frac{k}{n}\right) \right) r_k(x) + \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| \ge \delta}} \left(f(x) - f\left(\frac{k}{n}\right) \right) r_k(x) \right| \\
&\le \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| < \delta}} \underbrace{\left| f(x) - f\left(\frac{k}{n}\right) \right|}_{< \frac{\varepsilon}{2}} r_k(x) + \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| \ge \delta}} \underbrace{\left| f(x) - f\left(\frac{k}{n}\right) \right|}_{\le 2M} r_k(x) \\
&\le \frac{\varepsilon}{2} \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| < \delta}} r_k(x) + 2M \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| \ge \delta}} r_k(x) \\
&\le \frac{\varepsilon}{2} + 2M \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| \ge delta}} \overbrace{\frac{\left(x - \frac{k}{n}\right)^2}{\delta^2}}^{\ge 1} r_k(x) \\
&\le \frac{\varepsilon}{2} + \frac{2M}{\delta^2} \sum_{\substack{k \\ \left|x - \frac{k}{n}\right| \ge \delta}} \frac{(nx-k)^2}{n^2} r_k(x) \\
&\le \frac{\varepsilon}{2} + \frac{2M}{\delta^2 n^2} \sum_{k=1}^{n} (nx-k)^2 r_k(x)
\end{aligned}$$

Using (4), we arrive at the estimate

$$|f(x) - p_n(x)| < \frac{\varepsilon}{2} + \frac{2M}{n^2 \delta^2} n \underbrace{x(1-x)}_{\le 1/4}$$

because $x \in [0,1]$ and $\frac{1}{4}$ is the maximum of the parabola.

So we have the final estimate

$$|f(x) - p_n(x)| < \frac{\varepsilon}{2} + \frac{1}{2} \frac{M}{n\delta^2}$$

§26 Tuesday, October 22, 2013

§26.1 Bernstein polynomials

Review: $f \in [0,1]$ Bernstein polynomials

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

Let $\varepsilon > 0$, let $\delta > 0$ be such that $\forall x, y \in [0,1]$,

$$|x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$$

Let $M = \|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.
Then, $\forall x \in [0,1]$ and $n \geq 1$,

$$|f(x) - p_n(x)| < \frac{\varepsilon}{2} + \frac{M}{2n\delta^2}$$

Continuing the proof: Let $n(\varepsilon)$ be the smallest integer such that

$$n(\varepsilon) \geq \frac{M}{\varepsilon\delta^2}$$

Then, for $n \geq n(\varepsilon)$, $\frac{M}{2n\delta^2} \leq \frac{\varepsilon}{2}$. So for $n > n(\varepsilon)$, for all $x \in [0,1]$,

$$|f(x) - p_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, given f in $C([0,1])$, $\varepsilon > 0$, any Bernstein polynomial $p_n(x)$ with $n \geq n(\varepsilon)$ we have

$$\|f - p_n\|_\infty < \varepsilon$$

□

Remark. Suppose, in addition, f is Lipschitz on $[0,1]$. That is, $\exists K \geq 0$ such that $|f(x) - f(y)| \leq K|x-y|$, for all $x, y \in [0,1]$.

If f is continuous on $[0,1]$, differentiable on $(0,1)$, and $M = \sup_{x\in(0,1)} |f'(x)| < \infty$, then by mean value theorem, f is Lipschitz with $K = M+1$.

In the case of f Lipschitz, given $\varepsilon > 0$, if $\delta = \frac{\varepsilon}{2K}$ and $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq K|x-y| < \frac{\varepsilon}{2}$$

Then $n(\varepsilon) \geq \frac{M}{\varepsilon\left(\frac{\varepsilon}{2K}\right)^2} = \frac{M4K^2}{\varepsilon^3}$.

This is useful for numerics to know how big $n(\varepsilon)$ needs to be.

§26.2 Separability of $C([a,b])$ under p-norms

Proposition 26.1 $(C([0,1]), \|\cdot\|_\infty)$ is separable.

Proof. Let $\mathcal{P}_N$ be the collection of all polynomials of degree $\leq N$ whose coefficients are rational.

$$P_n = \{r_0 + r_1 x + \ldots + r_N x^N : r_k \in \mathbb{Q}\}$$

Let

$$f : P_N \to \mathbb{Q} \times \mathbb{Q} \times \ldots \times \mathbb{Q}$$

with $f(p) = (r_0, r_1, \ldots, r_N)$. f is a bijection, so $\mathcal{P}_N$ is countable.

Let $\mathcal{S} = \bigcup_{N=0}^{\infty} \mathcal{P}_N$. Countable union of countable sets is countable. We claim $\mathcal{S}$ is dense. Let $f \in [0,1]$ and $\varepsilon > 0$. Choose

$$p(x) = a_0 + a_1 + \ldots + a_n x^n$$

such that $\|f-p\| < \frac{\varepsilon}{2}$.

Choose rational numbers such that $|a_k - r_k| < \frac{\varepsilon}{2(n+1)}$.

Set $g(x) = r_0 + r_1 x + \ldots + r_n x^n$.

$$|p(x) - g(x)| = \left|\sum_{k=0}^{n} x^n (a_k - r_k)\right| \leq \sum_{k=0}^{n} x^n |a_k - r_k| < (n+1)\frac{\varepsilon}{2(n+1)} = \frac{\varepsilon}{2}$$

So, $\|p-g\|_\infty = \sup_{x\in[0,1]} |p(x)-g(x)| \leq \frac{\varepsilon}{2}$.

Hence, for $g \in \mathcal{S}$, $\|f-g\|_\infty \leq \|f-p\|_\infty + \|p-g\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Hence, $\mathcal{S}$ is dense. □

Proposition 26.2 Let $[a,b]$ be finite closed interval. Then polynomials are dense in $(C([a,b]), \|\cdot\|_\infty)$ and $(C([a,b]), \|\cdot\|_\infty)$ is separable.

Proof. Change of variables: define F by

$$F : C([0,1]) \to C([a,b])$$
$$F(f(x)) \to f\left(\frac{x-a}{b-a}\right)$$

for $x \in [a,b]$.

F is onto, $1:1$, and

$$\begin{aligned}\|F(f)\|_\infty &= \sup_{x\in[a,b]} \big|F(f)(x)\big| = \sup_{x\in[a,b]} \left|f\left(\frac{x-a}{b-a}\right)\right| \\ &= \sup_{t\in[0,1]} |f(t)| = \|f\|_\infty\end{aligned}$$

$F^{-1} : C([a,b]) \to C([0,1])$ acts as $(F^{-1}(g))(x) = g(a+(b-a)x)$ for $x \in [0,1]$. F maps polynomials to polynomials. Same for F^{-1}.

Let $g \in C([a,b])$ and $\varepsilon > 0$. Then $F^{-1}(g) \in C([0,1])$. Let p be a polynomial such that $\|F^{-1}(g) - P\|_\infty < \varepsilon$. Then $\|F(F^{-1}(g)) - F(p)\|_\infty = \|g - F(p)\|_\infty < \varepsilon$.

By exactly the same argument, if $\mathcal{S}$ is dense in $[0,1]$, the image $F(\mathcal{S})$ is dense in $C([a,b])$. If $\mathcal{S}$ is countable, so is $F(\mathcal{S})$ (because F is a bijection). Since $C([a,b])$ is separable, $C([a,b])$ is also separable. □

Proposition 26.3 $(C([a,b]), \|\cdot\|_p)$ is separable for $1 \leq p \leq \infty$.

Proof. $\|f\|_p = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}$.

$$\begin{aligned}\|f-g\|_p &= \left(\int |f(x)-g(x)|^p \,\mathrm{d}x\right)^{1/p} \\ &\leq \sup_{x\in[a,b]} |f(x)-g(x)|(b-a)^{1/p} \\ \implies \|f-g\|_p &\leq \|f-g\|_\infty (b-a)^{1/p}\end{aligned}$$

Given $\varepsilon > 0$, choose polynomial q such that
$\|f-q\|_p < \frac{\varepsilon}{(b-a)^{1/q}}$. So $\|f-q\|_p \leq \|f-q\|_\infty (b-a)^{1/p} < \varepsilon$.

Similarly, if $\mathcal{S}$ is dense in $(C([a,b]), \|\cdot\|_\infty)$, it is dense in $(C([a,b]), \|\cdot\|_p)$. So all these normed spaces are separable. □

§26.3 Separability and Product Spaces

If $(X, d_1), \ldots, (X_n, d_n)$ are finitely many separable metric spaces and (X, d) is their product, then (X, d) is separable.

Proposition 26.4 Let (X_k, d_k), $k = 1, 2, \ldots$ be a sequence of separable metric spaces. Let (X, d) be their product. Then (X, d) is separable.

Proof. Let $\bar{x}_k \in X_k$ be fixed points. Let S_k be a countable dense set in X_k. Define for any $N \geq 1$,

$$\tilde{S}_N = \{x = (x_k)_{k=1}^{\infty} : x_k \in S_k, 1 \leq k \leq N, x_n = \bar{x}_k \text{ if } k > N\}$$

A sequence $(x_n)_{n=1}^{\infty}$ belongs to $\tilde{S}_N$ if it is of the form

$$(s_1, s_2, \ldots, s_N, \bar{X}_{N+1}, \bar{X}_{N+2}, \ldots)$$

for $s_k \in X_k$. So $\tilde{S}_N$ is in bijective correspondence with $S_1 \times \ldots \times S_N$, so $\tilde{S}_N$ is countable.

Hence,

$$S = \bigcup_{N=1}^{\infty} \tilde{S}_N$$

is also countable. We claim S is dense in the product space (X, d).

Let $x = (x_k)_{n=1}^{\infty}$ be a given sequence and let $\varepsilon > 0$. Choose N such that $\sum_{k=N+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2}$. Choose $s_k \in S_k$, $1 \leq k \leq N$, such that $d_k(x_k, s_k) < \frac{\varepsilon}{2N}$.

Let now $y = (s_1, \ldots, s_N, \bar{x}_{N+1}, \bar{x}_{N+2}, \ldots)$. Then $y \in S$ and

$$\begin{aligned} d(x, y) &= \sum_{k=1}^{N} 2^{-n} \frac{d_n(x_n, s_n)}{1 + d_n(x_n, s_n)} + \sum_{k=N+1}^{\infty} 2^{-k} \overbrace{\left(\frac{d_k(x_n, \bar{x}_n)}{1 + d_n(x_n, \bar{x}_n)} \right)}^{\leq 1} \\ &\leq \sum_{k=1}^{N} d_n(x_k, s_k) + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

□

§27 Thursday, October 24, 2013

§27.1 Practice problem (with supremum)

Let X be a set and $f : X \to [0, \infty)$ a bounded function. Let $G : [0, \infty) \to [0, \infty)$ that is increasing (i.e. $G(y) \geq G(x)$ if $y \geq x$) and continuous.

Then $G\left(\sup_{x \in X} f(x)\right) = \sup_{x \in X} G(f(x))$.

Proof. $\sup_{x \in X} f(x)$ is the least upper bound of the set $\{f(x) : x \in X\}$. Denote it by M. $M \geq f(x)$ for all $x \in X$.

So for any $\varepsilon > 0$,

$$\exists x_\varepsilon \in X, M - \varepsilon < f(x_\varepsilon)$$

G is increasing, so $G(M) \geq G(f(x))$. Then $G(M)$ is an upper bound for the set $\{G(f(x)) : x \in X\}$. So $G(M) \geq \sup_{x \in X} G(f(x))$.

We also have that $G(M - \varepsilon) \leq G(f(x_\varepsilon)) \leq \sup_{x \in X} G(f(x))$. G is continuous at M, so take $\varepsilon \to 0$. Then $G(M) \leq \sup_{x \in X} G(f(x))$. □

§27.2 Midterm problem (2008)

Let $C_b(\mathbb{R})$ be the vector space of all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$.

$$\|f\| = \sup_{x\in\mathbb{R}} |f(x)| < \infty$$

1. Prove that $\|\cdot\|$ is a norm on $C_b(\mathbb{R})$
2. Prove that $(C_b(\mathbb{R}), \|\cdot\|)$ is a Banach space.
3. Prove that $(C_b(\mathbb{R}), \|\cdot\|)$ is not separable.

Exercise 27.1 Numbers 1 and 2

Proof of 3. Important: $C_b([a,b]) = C([a,b])$ which is separable. But $C_b(\mathbb{R})$ is not separable; one difference; $[a,b]$ is compact, while $\mathbb{R}$ is not. Note that $C_b(\mathbb{R})$ is crucial for probability.

It is enough to show that the product space (X, d), where

$$X = C_b(\mathbb{R}) \times C_b(\mathbb{R})$$

is not separable.

$F \in X$, $F = (f_1, f_2)$ where $f_1, f_2 \in C_b(\mathbb{R})$.

$$d(F,G) = \sqrt{\left(\sup_{x\in\mathbb{R}} \left|f_1(x) - g_1(x)\right|\right)^2 + \left(\sup_{x\in\mathbb{R}} \left|f_2(x) - g_2(x)\right|\right)^2}$$

Let $F_t = (\cos(tx), \sin(tx))$ where $t \in \mathbb{R}$ is a parameter, x is variable.

For $t \neq s$,

$$\begin{aligned}
d(F_t, F_s) &= \sqrt{\left(\sup_{x\in\mathbb{R}} |\cos(tx) - \cos(sx)|\right)^2 + \left(\sup_{x\in\mathbb{R}} |\sin(tx) - \sin(sx)|\right)^2} \\
&= \sqrt{\sup_{x\in\mathbb{R}} |\cos(tx) - \cos(sx)|^2 + \sup_{x\in\mathbb{R}} |\sin(tx) - \sin(ts)|^2} \\
&\geq \sqrt{\sup_{x\in\mathbb{R}} \left(|\cos(tx) - \cos(sx)|^2 + |\sin(tx) - \sin(sx)|^2\right)}
\end{aligned}$$

using $\sup(A) + \sup(B) \geq \sup(A+B)$

$$\begin{aligned}
&= \sup_{x\in\mathbb{R}} \sqrt{(\cos(tx) - \cos(sx))^2 + (\sin(tx) - \sin(sx))^2} \\
&= \sup_{x\in\mathbb{R}} \sqrt{\cos^2(tx) + \cos^2(sx) - 2\cos(tx)\cos(sx) + \sin^2(tx) + \sin^2(sx) - 2\sin(tx)\sin(sx)} \\
&= \sup_{x\in\mathbb{R}} \sqrt{2 - 2(\cos(tx)\cos(sx) + \sin(tx)\sin(sx)} \\
&= \sup_{x\in\mathbb{R}} \sqrt{2 - 2\cos((t-s)x)} \\
&\geq \sqrt{2 - 2\cos\left(\frac{(t-s)\pi}{2(t-s)}\right)} = \sqrt{2} > 1
\end{aligned}$$

Then $\left\{B\left(F_t, \frac{1}{2}\right)\right\}$ is a disjoint uncountable family of open balls in (X, d), so (X, d) is not separable. Thus, $C_b(\mathbb{R})$ is not separable.

□

§27.3 Chapter 10: Metric Spaces: Three Fundamental Theorems

§27.3.1 10.1: Arzela-Ascoli Theorem

X a compact metric space. $C(X)$ is the vector space of all continuous functions

$$f : X \to \mathbb{R}$$

f achieves its minimum and maximum value on X, and that f is uniformly continuous.

Set

$$\|f\| = \sup_{x \in X} |f(x)|$$

Proposition 27.2 $C(X)$ is a Banach space.

Remark. We have proven this in the case where X is an interval in $\mathbb{R}$; the general proof is the exact same, up to replacing $[a, b]$ by X.

Exercise 27.3 Check this

Important question: Describe compact subsets of $C(X)$.

$C(X)$ is infinite dimensional except in trivial cases (X finite, etc.). If $D \subset C(X)$ is compact, then D is closed and bounded. If $\dim C(X) = \infty$, then we need additional conditions to characterize compactness. That is what the A-A theorem achieves.

Theorem 27.4 ***(Arzela-Ascoli)*** *Let X be a compact metric space and $D \subset C(X)$. T.F.A.E.*

1. *D is compact in $(C(X), \|\cdot\|)$*
2. *D is closed and bounded* and

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in D)(\forall x, y \in X)(d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon) \qquad (\star)$$

Remark. This condition is called "equicontinuity". A family of functions satisfying $(\star)$ is called an equicontinuous family. Compare with uniform continuity:

$$(\forall \varepsilon > 0)(\forall f \in C(X))(\exists \delta_f > 0)(d(x, y) < \delta_f \implies |f(x) - f(y)| < \varepsilon)$$

The point of equicontinuity is that δ_f does not depend on f, if f is in an equicontinuous family.

Proof. (1) $\implies$ (2) (the "easy" side).

D is closed and bounded, since every compact set in any metric space is closed and bounded. We need to show that D is equicontinuous.

Fix $\varepsilon > 0$. Consider

$$\left\{ B\left(f, \frac{\varepsilon}{3}\right) \right\}_{f \in D}$$

This is an open cover of D, so there exists a finite subcover, that is, $f_1, \ldots, f_\ell \in D$ such that

$$D \subset B\left(f_1, \frac{\varepsilon}{3}\right) \cup \ldots \cup B\left(f_\ell, \frac{\varepsilon}{3}\right)$$

Each f_k is uniformly continuous. So there exists $\delta_k > 0$ such that $d(x, y) < \delta_k \implies |f_k(x) - f_k(y)| < \frac{\varepsilon}{3}$.

Set $\delta = \min_{1 \le k \le \ell} \delta_k > 0$.

The claim is that this δ will suffice for equicontinuity.

Let $f \in D$. Then, $f \in B\left(f_k, \frac{\varepsilon}{3}\right)$ for some $1 \le k \le \ell$. If $d(x, y) < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)| \\ &\le |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f(y) - f_k(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

where the first and third terms are less than $\frac{\varepsilon}{3}$ because f is in the ball around f_k of radius $\frac{\varepsilon}{3}$. The middle term is less than $\frac{\varepsilon}{3}$ by choice of δ.

□

§28 Friday, October 25, 2013

Recall:

Theorem 16.1 ***(Extension by uniform continuity)*** *Let (X, d_X) and (Y, d_Y) be metric spaces and suppose that Y is complete.*

Let $S \subset X$ be a dense set (this means $\operatorname{cl} S = X$) and $f : S \to Y$ a uniformly continuous function; this means

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in S)(d_X(x,) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$$

Then there exists a unique uniformly continuous function

$$\bar{f} : X \to Y$$

Theorem 27.4 ***(Arzela-Ascoli)*** *Let X be a compact metric space and $D \subset C(X)$. T.F.A.E.*

1. *D is compact in $(C(X), \|\cdot\|)$*
2. *D is closed and bounded* and

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in D)(\forall x, y \in X)(d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon) \qquad (\star)$$

§28.1 Proof of AA:

(1) $\implies$ (2): Last class

(2) $\implies$ (1):

Step 1: Since X is compact, X is separable. Let

$$S = \{x_j\}_{j=1}^{\infty}$$

be a countable dense set in X.

Step 2: D is bounded, hence $\exists M > 0$ such that $\forall f \in D$,

$$\|f\| = \sup_{x \in X} |f(x)| \leq M$$

Let now $(f_n)_{n=1}^{\infty}$ be a sequence of functions in D. We need to show that the sequence has a subsequence that converges in $(C(X), \|\cdot\|)$. Since D is closed, the limit of subsequence will be in D, which will imply D is compact.

Step 3: Consider sequences

$$(f_n(x_j))_{j=1}^{\infty}, n = 1, 2, 3, \ldots$$

These sequences are contained in the product space

$$\prod_{j=1}^{\infty} [-M, M]$$

If $z = (z_j)_{j=1}^{\infty}$ is an element of $\prod_{j=1}^{\infty} [-M, M]$, then

$$d(z, z') = \sum_{j=1}^{\infty} \frac{|z_j - z'_j|}{1 + |z_j - z'_j|}$$

$\prod_{j=1}^{\infty} [-M, M]$ is compact, as it is a product of compact metric spaces.

Hence, the sequences $(f_n(x_j))_{j=1}^{\infty}$, $n = 1, 2, \ldots$ has a convergent subsequence in the product space. Denote that subsequence by

$$(f_{n_k}(x_j))_{j=1}^{\infty}, k = 1, 2, \ldots$$

Recall that convergence in the product space is equivalent to component-wise convergence.

So for every j, the limit

$$\lim_{k\to\infty} f_{n_k}(x_j) \text{ exists}$$

Define

$$g(x_j) = \lim_{k\to\infty} f_{n_k}(x_j), j = 1, 2, \ldots$$

This defines g on $S = \{x_j\}_{j=1}^{\infty}$.

Step 4: g is uniformly continuous on S. Fix $\varepsilon > 0$. By equicontinuity, $\exists \delta > 0$ such that for all $x_i, x_j \in S$,

$$d(x_i, x_j) < \delta \implies |f_{n_k}(x_i) - f_{n_k}(x_j)| < \frac{\varepsilon}{2}$$

This is true for all k, so taking $k \to \infty$ gives

$$d(x_i, x_j) < \delta \implies |g(x_i) - g(x_j)| \leq \frac{\varepsilon}{2} < \varepsilon$$

So g is uniformly continuous. $\mathbb{R}$ is complete and S is dense in X. Extension by uniform continuity gives that g extends uniquely to a uniformly continuous function

$$g : X \to \mathbb{R}$$

Step 5: $\forall x \in X$,

$$\lim_{k\to\infty} f_{n_k}(x) = g(x)$$

We know, so far, that this is true only for $x \in S$. Fix $x \in X$ and let $\varepsilon > 0$.

g is uniformly continuous, hence $\exists \delta_1 > 0$ such that for all $z', z'' \in X$, $d(z', z'') < \delta_1 \implies |g(z') - g(z'')| < \frac{\varepsilon}{3}$.

Equicontinuity implies there exists $\delta_2 > 0$ such that $\forall k$,

$$d(z', z'') < \delta_2 \implies |f_{n_k}(z') - f_{n_k}(z'')| < \frac{\varepsilon}{3}$$

Let $\delta = \min(\delta_1, \delta_2)$.

Since S is dense, $\exists\, j$ such that

$$d(x, x_j) < \delta$$

Then,

$$\begin{aligned}
|g(x) - f_{n_k}(x)| &= |g(x) - g(x_j) + g(x_j) - f_{n_k}(x_j) + f_{n_k}(x_j) - f_{n_k}(x)| \\
&\leq |g(x) - g(x_j)| + |g(x_j) - f_{n_k}(x_j)| + |f_{n_k}(x_j) - f_{n_k}(x)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |f_{n_k}(x_j) - g(x_j)|
\end{aligned}$$

Since $\lim_{k\to\infty} f_{n_k}(x_j) = g(x_j)$, there exists $K > 0$ such that for $k \geq K$, $|f_{n_k}(x) - g(x_j)| < \frac{\varepsilon}{3}$.

So for $k \geq K$, $|g(x) - f_{n_k}(x)| < \varepsilon$.

It may seem that we are done, but x_j depends on x, so K depends on x. We have proven pointwise convergence,

$$\lim_{k\to\infty} f_{n_k}(x) = g(x)$$

but not uniform convergence.

We have only used that X is separable, not that X is compact.

Step 6: $f_{n_k} \to f$ in $(C(X), \|\cdot\|_\infty)$, that is,

$$\lim_{k\to\infty} \|f_{n_k} - g\| = \lim_{k\to\infty} \sup_{x\in X} |f_{n_k}(x) - g(x)| = 0$$

Let $\varepsilon > 0$. Let δ be constructed as in Step 5:

g uniformly continuous, so we can find $\delta_1 > 0$ such that $d(z', z'') < \delta_1 \implies |g(z') - g(z'')| < \frac{\varepsilon}{3}$, and by equicontinuity, $\exists \delta_2 > 0$ such that $\forall k$, $d(z', z'') < \delta_2 \implies |f_{n_k}(z') - f_{n_k}(z'')| < \frac{\varepsilon}{3}$. We chose

$$\delta = \min(\delta_1, \delta_2)$$

Consider $\{B(x_n, \delta\}_{n=1}^{\infty}$. These balls cover X, since S is dense in X.

If $x \in X$, $B(x, \delta)$ has non-empty intersection with X, so $\exists j$ such that $x_j \in B(x, \delta)$, and so $x \in B(x_j, \delta)$.

Since X is compact, we have a finite subcover to this open cover.

$$X = B(x_i, \delta) \cup \ldots \cup B(x_{i_\ell}, \delta)$$

$$\lim_{k\to\infty} f_{n_k}(x_{i_r}) = g(x_{i_r}), \qquad 1 \leq r \leq \ell$$

Since ℓ is finite, $\exists K > 0$ such that

$$|f_{n_k}(x_{i_r} - g(x_{i_r})| < \frac{\varepsilon}{3}$$

for $k \geq K$ and all $r = 1, 2, \ldots, \ell$.

Let now $x \in X$. Then $x \in B(x_{i_r}, \delta)$ for some r, $1 \leq r \leq \ell$.

$$|g(x) - f_{n_k}(x)| \leq |g(x) - g(x_{i_r})| + |g(x_{i_r}) - f_{n_k}(x_{i_r})| + |f_{n_k}(x_{i_r}) - f_{n_k}(x)|$$

By choice of δ, $|g(x) - g(x_{i_r})| < \frac{\varepsilon}{3}$ and $|f_{n_k}(x_{i_r}) - f_{n_k}(x)| < \frac{\varepsilon}{3}$.

However, now by the choice of x_{i_r}, we have

$$|g(x_{i_r}) - f_{n_k}(x_{i_r})| < \frac{\varepsilon}{3}$$

So, $|g(x) - f_{n_k}(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$ for $k \geq K$ and K does not depend on x.

So, for $k \geq K$, $\|f_{n_k} - g\| < \varepsilon$. This proves the theorem.

□

§29 Friday, October 2013 - Tutorial VI

Let $\mathcal{C} = \{x = (x_n)_{n=1}^{\infty} \in \ell^\infty(\mathbb{N}) : \lim_{n\to\infty} |x_n| = 0\}$.

The closure with respect $\|\cdot\|_\infty$, $\mathrm{cl}(\ell'(\mathbb{N})) = \mathcal{C}$.

Let X be the collection of all sequence whose variation is finite and limit goes to zero:

$$X := \{x \in \mathcal{C} : \sum_{n=1}^{\infty} |x_{n+1} - x_n| < \infty\}$$

where

$$\|x\| = \sum_{n=1}^{\infty} |x_{n+1} - x_n|$$

is is easily shown to be a norm on X.

Claim. $(X, \|\cdot\|)$ *is a Banach space.*

Proof. Let $(x^{(i)})_{i=1}^{\infty}$ be a Cauchy sequence in X.

Then $\forall \varepsilon > 0$, there exists N such that $\forall n, m \geq N$,

$$\|x^{(n)} - x^{(m)}\| = \sum_{i=1}^{\infty} |x_{i+1}^{(n)} - x_{i+1}^{(m)} - (x_i^{(n)} - x_i^{(m)})| < \varepsilon$$

Pick some $i \in \mathbb{N}, M \in \mathbb{N}$ and consider $\{i, \ldots, i+M\}$. Then we get

$$|x_{i+M}^{(n)} - x_{i+M}^{(m)} - (x_i^{(n)} - x_i^{(m)})| \leq \sum_{j=i}^{M} |x_{j+1}^{(n)} - x_{j+1}^{(m)} - (x_j^{(n)} - x_j^{(m)})| < \varepsilon$$

SO, $|x_i^{(n)} - x_i^{(m)}| < \varepsilon$, for all i. The sequence of reals $(x_i^{(n)})_{n=1}^{\infty}$ converges to a point x_i for each i. The convergence is uniform as well (same N).

For all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, and $\forall i \in \mathbb{N}$

$$|x_i^{(n)} - x_i| < \varepsilon$$

Then

$$\lim_{i \to \infty} |x_i^{(n)} - x_i| = \lim_{i \to \infty} |x_i| < \varepsilon \to 0$$

Let $x = (x_i)_{i=1}^{\infty}$. Pick $m \in \mathbb{N}$.

$$\begin{aligned}
\|x - x^{(m)}\| &= \sum_{i=1}^{\infty} |x_{i+1} - x_{i+1}^{(m)} - (x_i - x_i^{(m)}| \\
&= \sum_{i=1}^{\infty} \liminf_{n \to \infty} |x_{i+1}^{(n)} - x_{i+1}^{(m)} - (x_i^{(n)} - x_i^{(m)})| \\
&\leq \liminf_{n \to \infty} \sum_{i=1}^{\infty} |x_{i+1}^{(n)} - x_{i+1}^{(m)} - (x_i^{(n)} - x_i^{(m)})|
\end{aligned}$$

so, for $m \geq N$,

$$\leq \liminf_{n \to \infty} \varepsilon = \varepsilon$$

□

Claim. $(X, \|\cdot\|)$ *is a separable.*

Proof.

$$A_n = \{x = (x_i)_{i=1}^{\infty} : x_i = 0 \, \forall i > n, x_i \in \mathbb{Q} \; \forall i \leq n\}$$

$A = \bigcup_{n=1}^{\infty} A_n$. A is countable.

Let $x \in X$. Choose $N_1 > 0$ such that

$$\sum_{i=N+1}^{i} nfty|x_{i+1} - x_i| < \frac{\varepsilon}{2}$$

Pick $a \in A_N$ such that $a_0 = 0$, a_1 satisfies

$$|x_1 - a_1| < \frac{\varepsilon}{2}$$

If a_i is defined, choose a_{i+1} so that

$$|a_{i+1} - a_i - (x_{i+1} - x_i)| < 2^{-i} \frac{\varepsilon}{2}$$

do this until $a_1, \ldots, a_N$ are defined.

$$\begin{aligned} a_x\| &= \sum_{i=1}^{\infty} |x_{i+1} - a_{i+1} - (x_i - a_i)| \\ &= \sum_{i=1}^{N} |x_{i+1} - x_i - (a_{i+1} - a_i)| + \sum_{i=N+1}^{\infty} |x_{i+1} - x_i| \end{aligned}$$

$$< \sum_{i=1}^{N} 2^{-i} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Next time: we will explore the closure and interior.
Let $a = (a_n)_{n=1}^{\infty}$ be a sequence of real numbers.
Let $B_a = \{x \in \ell^{\infty}(\mathbb{N}) : |x_n| \leq |a_n|\}$ for all $n \in \mathbb{N}$.

Claim. T.F.A.E.

1. *B_a is compact in $\ell^{\infty}(\mathbb{N})$*
2. $\lim_{n\to\infty} a_n = 0$

Proof. $\neg(2) \implies \neg(1)$
$\lim_{n\to\infty} a_n \neq 0 \implies \exists \varepsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists n \geq N$ such that $|a_n| \geq \varepsilon$.
$\implies \exists (a_{n_k})_{k=1}^{\infty}$ a subsequence of a such that $|a_{n_k}| \geq \varepsilon$ for all $k \in \mathbb{N}$.
For each n_k, define

$$x^{(n_k)} = (x_j^{(n_k)})_{j=1}^{\infty}$$

$$x_j^{(n_k)} = \begin{cases} \varepsilon & \text{if } j = n_k \\ \\ 0 & \text{if otherwise} \end{cases}$$

So,

$$|x_j^{(n_k)}| \leq |a_j| \quad \forall j$$

So, $x^{(n_k)} \in B_a$, for all $k \in \mathbb{N}$.
Then $(x^{(n_k)})_{k=0}^{\infty}$ is a sequence in B_a.
So, by definition of $x^{(n_k)}$, $\|x^{(n_k)} - x^{(n_j)}\|_{\infty} = \varepsilon$ for all $j \neq k$. So there can be no convergent subsequences.
So B_a cannot be compact.
Now, to show that (2) $\implies$ (1): Suppose $\lim_{n\to\infty} |a_n| = 0$.
Let $(x^{(k)})_{k=1}^{\infty}$ be a sequence with $x^{(k)} \in B_a$. Consider, for fixed n, $(x_n^{(k)})_{k=1}^{\infty}$ as an element of $\prod_{k\in\mathbb{N}}[-|a_n|, |a_n|]$. This space is compact with respect to the product metric

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d(x_k, y_k)}{1 + d(x_k, y_k)}$$

So $x^{(k)}$ has a convergent subsequence $x^{(k_j)}$ which converges to $y \in \ell^{\infty}$. So, moreover, for each $n \in \mathbb{N}$,

$$\lim_{j\to\infty} x_n^{k_j} = y_n$$

Each $|x_n^{(k_j)}| \leq |a_n|$. So $|y_n| \leq |a_n|$, so $y \in B_a$.
Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$\sup_{n\geq N} |a_n| < \frac{\varepsilon}{3}$$

Since $N < \infty$, $\exists J \in \mathbb{N}$ such that for all $j \geq J$,

$$\max_{1 \leq n \leq N-1} |x_n^{(k_j)} - y_n| < \frac{\varepsilon}{3}$$

$$\begin{aligned}\|x^{(k_j)} - y\|_\infty = \sup_{n\geq 1} |x_n^{(k_j)} - y_n| &\leq \max_{1\leq n \leq N-1} |x_n^{(k_j)} - y_n| + \sup_{n \geq N} |x_n^{(k_j)} - y_n| \\ &\leq \frac{\varepsilon}{3} + \sup_{n\geq N} |x_n^{(k_j)}| + \sup_{n \geq N} |y_n| \\ &\leq \frac{\varepsilon}{3} + 2\sup_{n \geq N} |a_n| \leq \varepsilon\end{aligned}$$

□

Claim. Let $X \subset \ell^p(\mathbb{N})$. Then X is compact if and only if the following conditions hold:

1. *X is closed and bounded*
2. $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall x \in X)\left(\sum_{n=N}^\infty |x_n|^p < \varepsilon\right)$

Proof. Suppose X is compact. We know any compact subset is closed and bounded. Given $\varepsilon > 0$, choose $x^{(1)}, \ldots, x^{(n)}$ such that

$$X \subset \bigcup_{k=1}^n B\left(x^{(k)}, \frac{\varepsilon}{2}\right)$$

for each k, choose N_k such that

$$\sum_{j=N_k}^\infty |x_j^{(k)}|^p < \left(\frac{\varepsilon}{2}\right)^p$$

Consider $N = \max_{1\leq k \leq n} N_k < \infty$. Given $x \in X$,

$$x \subset B\left(x^{(k)}, \frac{\varepsilon}{2}\right)$$

for some $1 \leq k \leq n$.

$$\left(\sum_{j=N}^\infty |x_j|^p\right)^{1/p} \leq \left(\sum_{j=N}^\infty |x_j - x_j^{(k)}|^p\right)^{1/p} + \left(\sum_{j=N}^\infty |x_j^{(k)}|^p\right)^{1/p} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

where the first term is bounded because the two sequences are in the same ball, and the second term is bounded by choice of N.

Now, we want to prove the converse. Suppose X satisfies (1) and (2). Let $(x^{(k)})_{k=1}^\infty$, $x^{(k)} \in X$ for all k.

Let $M > 0$ such that $\forall x \in X$, $\|x\|_p \leq M$.

Consider then $(x_n^{(k)})_{k=1}^\infty$ as an element of $\prod_{n\in\mathbb{N}}[-M, M]$ which is compact.

Then $x_n^{k_j} \to y_n$ for each n.

WTS: $y = (y_n)_{n=1}^\infty \in \ell^p$.

For finite N, $\left(\sum_{n=1}^\infty |x_n^{(k_j)}|^p\right)^{1/p} \leq M$. Take $j \to \infty$ to get $\left(\sum_{n=1}^N |y_n|^p\right)^{1/p} \leq M$.

This implies that $\|y\|_p \leq M < \infty$ so $y \in \ell^p(\mathbb{N})$.

Last thing to show: $\|x^{(k_j)} - y\|_p \to 0$ as $j \to \infty$.

First choose $N_1 \in \mathbb{N}$ such that

$$\left(\sum_{n=N_1} |y_n|^p\right)^{1/p} < \frac{\varepsilon}{3}$$

and N_2 such that $\forall x \in X$

$$\left(\sum_{n=N_2}^{\infty} |x_n|^p\right)^{1/p} < \frac{\varepsilon}{3}$$

Take $N = \max\{N_1, N_2\}$. Finally, choose $J \in \mathbb{N}$ such that for all $j \geq J$,

$$\left(\sum_{n=1}^{N-1} |x_n^{(k_j)} - y_n|^p\right)^{1/p} < \frac{\varepsilon}{3}$$

using that this is a finite sum.

Now, letting $j \geq J$,

$$\|x^{(k_j)} - y\|_p = \left(\sum_{n=1}^{\infty} |x_n^{(k_j)} - y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{N-1} |x_n^{(k_j)} - y_n|^p\right)^{1/p} + \left(\sum_{n=N}^{\infty} |y_n|^p\right)^{1/p}$$
$$\leq \frac{\varepsilon}{3} + \left(\sum_{n=N}^{\infty} |x_n^{(k_j)}|^p\right)^{1/p} + \left(\sum_{n=N}^{\infty} |y_n|^p\right)^{1/p} \leq \varepsilon$$

□

§30 Tuesday, October 29, 2013

§30.1 Arzela-Ascoli Theorem

Theorem 27.4 ***(Arzela-Ascoli)*** *Let X be a compact metric space and $D \subset C(X)$. T.F.A.E.*

1. *D is compact in $(C(X), \|\cdot\|)$*
2. *D is closed and bounded* and

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in D)(\forall x, y \in X)(d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon) \qquad (\star)$$

Looking at the proof of (2) $\implies$ (1), we notice we didn't use compactness of X until step 6. Steps 1-5 yield the following result:

Theorem 30.1 *Let X be a separable metric space and $f_n : X \to \mathbb{R}$ a sequence of continuous functions. Supppose the following:*

1. *The sequence f_n is bounded, that is, $\exists M > 0$ such that for all n,*

$$\sup_{x \in X} |f_n(x)| \leq M$$

2. *The sequence f_n is equicontinuous, that is,*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall n \in \mathbb{N})(d(x,y) < \delta \implies |f_n(x) - f_n(y)| < \varepsilon)$$

Then there exists a subsequence f_{n_k} such that for all $x \in X$, the limit

$$\lim_{k \to \infty} f_{n_k}(x) = g(x)$$

exists and moreover, the limiting function $g(x)$ is uniformly continuous on X.

We can also reformulate A-A:

Corollary 30.2 *(of A-A)* Let X be a compact metric space and $D \subset C(X)$. T.F.A.E.:

1. $\text{cl}(D)$ is compact in $(C(X), \|\cdot\|)$
2. D is bounded and equicontinuous

Proof. (1) $\implies$ (2)**:** If $\text{cl}(D)$ is compact, then by A-A, $\text{cl}(D)$ is bounded and equicontinuous. Since $D \subseteq \text{cl}\, D$, D is also bounded and equicontinuous.

(2) $\implies$ (1)**:** Suppose D is bounded and equicontinuous. Then $\text{cl}(D)$ is closed; we need to show bounded and equicontinuious.

[Boundedness:] Let $M > 0$ such that $\forall f \in D$, $\|f\| \leq M$. Let $g \in \text{cl}(D)$ and f_n be a sequence in D such that $\lim_{n\to\infty} \|f_n - g\| = 0$. Then $\|g\| = \|g_- f_n + f_n\| \leq \|g - f_n\| + \|f_n\| \leq \|g - f_n\| + M \leq M$, taking $n \to \infty$.

[Equicontinuity:] We need to show that given $\varepsilon > 0$, $\exists \delta > 0$ such that $\forall g \in \text{cl}\, D$,

$$d(x,y) < \delta \implies |g(x) - g(y)| < \varepsilon$$

We know D is equicontinuous. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $\forall f \in D$,

$$d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon$$

This δ will serve for $\text{cl}(D)$ too.

Let $g \in \text{cl}\, D$. Let $f \in D$ be such that

$$\|g - f\| = \sup_{x\in X} |g(x) - f(x)| < \frac{\varepsilon}{3}$$

Suppose that $d(x,y) < \delta$. Then,

$$|g(x)-g(y)| \leq |g(x)-f(x)+f(x)-f(y)+f(y)-g(y)| \leq \|g-f\|+|f(x)-f(y)|+\|f-g\| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

□

Example 30.3 Let $X = [0,1]$ and let k_1, k_2 be two positive constants. Let $D \subseteq C([0,1])$ be a collection of functions such that

1. $\forall f \in D$,
$$\sup_{x\in[0,1]} |f(x)| \leq k_1$$
2. Each $f \in D$ is differentiable on the open interval $(0,1)$ and for all $f \in D$,
$$\sup_{x\in[0,1]} |f'(x)| \leq k_2$$

Then $\text{cl}\, D$ is compact on $(C(X), \|\cdot\|)$.

Proof. We need to show that D is bounded and equicontinuous (then $\text{cl}\, D$ is compact by A-A (corollary)). (1) implies that D is bounded.

By the mean value theorem, for $x, y \in [0,1]$, $\exists c \in (0,1)$ such that

$$f(x) - f(y) = f'(c)(x-y)$$

SO,

$$|f(x) - f(y)| = |f'(c)||x-y| \leq k_2|x-y|$$

Fix $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{k_2}$.
Then, if $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq k_2|x-y| < k_2\delta = \varepsilon$$

This is true for all $f \in D$, so we have equicontinuity.

□

Example 30.4 Let $X = [0,1]$, k_1, k_2 positive constants. Let $D \subseteq C([0,1])$ be a collection of functions such that

1. $|f(0)| \leq k_1 \ \forall f \in D$,
2. $\forall f \in D$, f is continuously differentiable on $[0,1]$ and

$$\int_0^1 |f'(x)|^2 \, \mathrm{d}x \leq k_2$$

Then $\mathrm{cl}(D)$ is compact in $C([0,1])$.

Proof. We need to show D is bounded and equicontinuous.

Bounded:

$$f(x) - f(0) + f(0) = \int_0^x f'(t) \, \mathrm{d}t + f(0) \implies f(x) = \int_0^x f'(t) \, \mathrm{d}t + f(0)$$

Then,

$$|f(x)| \leq \int_0^x |f'(t)| \, \mathrm{d}t + |f(0)| \leq \int_0^1 |f'(t)| \, \mathrm{d}t + |f(0)| \leq \left(\int_0^1 \mathrm{d}t \right)^{1/2} \left(\int_0^1 |f'(t)|^2 \, \mathrm{d}t \right)^{1/2} + |f(0)| \leq \sqrt{k_2} + k_1$$

Equicontinuity: let $x < y$.

$$\begin{aligned} f(x) - f(y) &= \int_x^y f'(t) \, \mathrm{d}t \\ |f(x) - f(y)| &\leq \int_x^y |f'(t)| \, \mathrm{d}t \leq \left(\int_x^y 1 \, \mathrm{d}t \right)^{1/2} \left(\int_x^y |f'(t)|^2 \, \mathrm{d}t \right)^{1/2} \\ &\leq \sqrt{|x-y|} \left(\int_0^1 |f'(t)|^2 \, \mathrm{d}t \right)^{1/2} \\ &\leq \sqrt{|x-y|} \sqrt{k_2} \end{aligned}$$

Same if $y < x$. Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon^2}{k_2}$.
Then, for all $f \in D$, if $|x - y| < \delta$,

$$|f(x) - f(y)| \leq \sqrt{|x-y|}\sqrt{k_2} < \sqrt{\delta}\sqrt{k_2} = \varepsilon$$

□

§31 Thursday, October 31, 2013

§31.1 Baire Category Theorem

Theorem 31.1 ***(Baire Category Theorem)*** *Let X be a complete metric space and $\{O_n\}_{n=1}^\infty$ be a countable collection of open and dense sets in X. Then*

$$\bigcap_{n=1}^\infty O_n$$

is dense in X.

§31.1.1 Proof

Recall that a set $S \subset X$ is dense if and only if for any non-empty open set $O \subset X$, $O \cap S \neq \varnothing$. So we need to show that given a non-empty open set $O \subset X$,

$$\left(\bigcap_{n=1}^{\infty} O_n\right) \cap O \neq \varnothing$$

The proof is based on the following construction:

Step 1: Consider

$$O_1 \cap O$$

since O_1 is dense, $O_1 \cap O \neq \varnothing$.

Obviously, $O_1 \cap O$ is open. Pick $x_1 \in O_1 \cap O$. Let $\varepsilon > 0$ be such that the $B(x_1, \varepsilon) \subset O_1 \cap O$.

Let $r_1 = \min\left(\frac{\varepsilon}{2}, 1\right)$. Then

$$\operatorname{cl} B(x_1, r_1) \subset \left\{y : d(x_1, y) \leq \frac{\varepsilon}{2}\right\} \subset B(x_1, \varepsilon) \subset O_1 \cap O$$

At the end of step 1, there exists x_1 and $0 < r_1 \leq 1$ such that

$$\operatorname{cl}(B(x_1, r_1)) \subset O_1 \cap O$$

Step 2: Consider

$$O_2 \cap B(x_1, r_1)$$

This is again an open set.

Pick $x_2 \in O_2 \cap B(x_1, r_1)$ and $\varepsilon > 0$ such that $B(x_2, \varepsilon) \subset O_2 \cap B(x_1, r_1)$.

Set

$$r_2 = \min\left(\frac{\varepsilon}{2}, \frac{1}{2}\right)$$

where the 2 of $\frac{1}{2}$ refers to Step 2.

$$\operatorname{cl} B(x_2, r_2) \subset \left\{y : d(x_2, y) \leq \frac{\varepsilon}{2}\right\} \subset O_2 \cap B(x_2, r_2)$$

End of step 2: there exists x_2 and $0 < r_2 \leq \frac{1}{2}$, such that

$$\operatorname{cl}(B(x_2, r_2)) \subset O_2 \cap B(x_1, r_1)$$

We continue inductively and construct a sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of X and a sequence of a real numbers $r_1, r_2, \ldots, r_n, \ldots$ such that $0 < r_n \leq \frac{1}{n}$ and

$$\operatorname{cl}(B(x_n, r_n)) \subset O_n \cap B(x_{n-1}, r_{n-1})$$

for $n \geq 2$, and

$$\operatorname{cl}(B(x_1, r_1)) \subset O_1 \cap O$$

Claim: The sequence x_n is Cauchy. Let $\varepsilon > 0$. Choose integer $N > 0$ such that

$$\frac{2}{N} < \varepsilon$$

Let $n, m \geq N$. Then by construction, we know

$$B(x_m, r_m) \subset B(x_N, r_N) \qquad B(x_n, r_n) \subset B(x_N, r_N)$$

In particular, $x_m, x_n \in B(x_N, r_N)$.

So,

$$d(x_m, x_n) \leq d(x_m, x_N) + d(x_N, x_m) \leq r_N + r_N \leq \frac{1}{N} + \frac{1}{N} < \varepsilon$$

by definition of r_N and then choice of N.

Since X is complete, $\lim_{n\to\infty} x_n = x$ exists.

Claim: $x \in \mathrm{cl}(B(x_n, r_n))$ for any n.

Fix n. Then for $m > n$, $B(x_m, r_m) \subset B(x_n, r_n) \subset \mathrm{cl}(B(x_n, r_n))$.

So the sequence $(x_k)_{k=n}^{\infty}$ is contained in the closed set $\mathrm{cl}(B(x_n, r_n))$.

So, $x = \lim_{n\to\infty} x_k$ is also contained in $\mathrm{cl}(B(x_n, r_n))$.

We have $x \in \mathrm{cl}(B(x_n, r_n)) \subset O_n \cap B(x_{n-1}, r_{n-1})$ for $n \geq 2$.

We then have $x \in O_n$ for $n \geq 2$. We also have $x \in \mathrm{cl}(B(x_1, r_1)) \subset O_1 \cap O$. Then $x \in O_1$ and $x \in O$.

Thus, we have

$$x \in \left(\bigcap_{n=1}^{\infty} O_n\right) \cap O$$

□

Baire Category Theorem *fails* if X is not complete.

Example 31.2 Let $X = \mathbb{Q}$, the rationals, and $d(x, y) = |x - y|$. $\mathbb{Q} = \{r_1, r_2, \ldots, r_n, \ldots\}$ an enumeration of the rationals.

Set $V_n = \mathbb{Q} \setminus \{r_n\}$. Then V_n is open and dense in $\mathbb{Q}$ (exercise).

Consider

$$\bigcap_{n=1}^{\infty} V_n$$

For any n, r_n is not in this intersection, so it is empty.

$$\bigcap_{n=1}^{\infty} V_n = \varnothing$$

Corollary 31.3 Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ be a countable sequence of closed sets in X.

Suppose that $\bigcup_{n=1}^{\infty} F_n$ has a non-empty interior. Then for at least one n, F_n has non-empty interior.

Proof. Suppose the statement is false.

Let $\{F_n\}_{n=1}^{\infty}$ be closed sets such that each F_n has empty interior but

$$\bigcup_{n=1}^{\infty} F_n$$

has a non-empty interior.

Let $O_n = F_n^c$. Then O_n is open and dense (why?).

If V is non-empty and open, then then $V \cap O_n \neq \varnothing$ otherwise $V \subset O_n^c = F_n$ which contradicts the assumption that F_n has empty interior.

Since $\bigcup_{n=1}^{\infty} F_n$ has non-empty interior, there is a non-empty open set O such that $O \subset \bigcup_{n=1}^{\infty} F_n$.

Then

$$\varnothing = O \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c = O \cap \left(\bigcap_{n=1}^{\infty} F_n^c\right) = O \cap \left(\bigcap_{n=1}^{\infty} O_n\right)$$

where $O_n = F_n^c$ and by the Baire Category Theorem, $\bigcap_{n=1}^{\infty} O_n$ is dense in X. Contradiction. □

§31.2 Some definitions

Definition 31.4 A set $A \subset X$ is called nowhere dense if its closure $\mathrm{cl}(A)$ has an empty interior.

Definition 31.5 A set $A \subset X$ is called a first category if it is a countable union of nowhere dense sets. Any other set is of second category.

In this language then, the Baire Category Theorem says that a complete metric space is itself a set of second category.

This immediately follows from the corollary to the BCT: Suppose $X = \bigcup_{n=1}^{\infty} A_n$ where A_n is nowhere dense. Then

$$X = \bigcup_{n=1}^{\infty} \mathrm{cl}(A_n)$$

X has a non-empty interior. Then for some n, $\mathrm{cl}(A_n)$ has a non-empty interior. This contradicts the assumption that A_n is nowhere dense. So X is a set of second category.

Definition 31.6 A set $A \subset X$ is called G_δ if A is a countable intersection of open sets.

Example 31.7 Points are G_δ sets: for $x \in X = \mathbb{R}^n$, take $O_n = B\left(x, \frac{1}{n}\right)$. Then $\bigcap_{n=1}^{\infty} O_n = \{x\}$.

Definition 31.8 A set is called an $F\sigma$-set if it is a countable union of closed sets.

Proposition 31.9 Let X be a countable metric space and $\{V_n\}_{n=1}^{\infty}$ a countable collection of dense G_δ-sets. Then

$$\bigcap_{n=1}^{\infty} V_n$$

is also a dense G_δ set.

§32 Friday, November 1, 2013

Proposition 32.1 Let X be a complete metric space, then a countable intersection of dense open G_δ sets

$$\bigcap_{n=1}^{\infty} V_n$$

is also dense and G_δ.

Proof. Let $\{V_k\}_{k=1}^{\infty}$ be a countable collection of dense G_δ sets. Since V_n is G_δ, then there are open sets $O_{n,m}$, $m = 1, 2, \ldots$ such that

$$V_n = \bigcap_{m=1}^{\infty} O_{n,m}$$

Now, since $V_n \subset O_{n,m}$ and V_n is dense, then $O_{m,n}$ is dense. Note

$$\bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} \left(\bigcap_{m=1}^{\infty} O_{n,m} \right) = \bigcap_{n,m=1}^{\infty} O_{n,m}$$

and $\{O_{n,m}\}_{n,m=1}^{\infty}$ is a countable collection of open dense sets. Thus, $\bigcap_{n=1}^{\infty} V_n$ is G_δ and dense by the Baire Category Theorem (B-C-T). ☐

Definition 32.2 Let X be a metric space. A point $x \in X$ is called isolated if $\exists \varepsilon > 0$ such that $B(x, \varepsilon) = \{x\}$.

Example 32.3 Let X be any set and d the discrete metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Then any point of X is isolated. Take $\varepsilon = 1$, $\quad B(x, y) = \{y \in X : d(x, y) < 1\} = \{x\}$.

Proposition 32.4 Let X be a complete metric space, then a countable dense set $S \subset X$ cannot be a G_δ set.

Consequence (B-C-T): If $O_n, \quad n = 1, 2, ...$ are open dense, then

$$\bigcap_{n=1}^{\infty} O_n$$

is G_δ and dense. So if X has no isolated points, $\bigcap_n O_n$ is countable.

Proof. Suppose that the statement were not true. Let $S = \{x_1, x_2, .., x_n\}$ be a countable dense set that is also G_δ.

$$S = \bigcap_{n=1}^{\infty} V_n$$

where V_n is open in X. Set $W_n = V_n - \{x_n\}$.

Claim. W_n is non-empty

Proof. If $W_n \neq \varnothing$ then $V_n = \{x_n\}$ and since V_n is open then $\exists \varepsilon > 0$ such that $B(x_n, \varepsilon) = \{x_n\}$. But then x_n would be an isolated point. □

Claim. W_n is open.

Proof. Let $x \in W_n$, then $x \in V_n$ and $x \neq x_n$. Since V_n is open, $\exists \varepsilon > 0$ such that $B(x_n, \varepsilon) \subset V_n$. Let $\varepsilon' = \min(\varepsilon, d(x, x_m)) > 0$. Then, $x_n \notin B(x, \varepsilon')$ and $B(x, \varepsilon') \subset B(x, \varepsilon) \subset V_k$. Thus, $B(x, \varepsilon') \subset W_k$ and W_k is open. □

Claim. W_n is dense.

Proof. Exercise. □

Now, note the W_n are open, dense sets and

$$\bigcap_{n=1}^{\infty} W_n \subset \bigcap_{n=1}^{\infty} V_n = S$$

But $W_n \not\ni x_n \ \forall n$. So, $x_n \notin \bigcap_n W_n \ \forall n$. Note

$$S \cap \left(\bigcap_{n=1}^{\infty} W_n \right) = \varnothing \quad \text{and} \quad \bigcap_{n=1}^{\infty} W_n \subset S$$

hence we must have $\bigcap_{n=1}^{\infty} W_n = \varnothing$, but this contradict the Baire Category Theorem.

□

§32.1 Consequences of B-C-T

1. Consider $C([a, b])$ with the usual sup norm. Consider $f \in C([a, b])$ which are not differentiable at any point $x \in (a, b)$. Such functions are called singular. B-C-T: $\exists$ a G_δ dense set $E \subset C([a, b])$ that consists only of singular functions.

   Quasi-periodic systems, systems far from equillibrium, modern theory of dynamical systems

   Probability $\implies$ singular functions.

2. Fourier series

   $C([-\pi, \pi])$, $f \in C([-\pi, \pi])$, $f(-\pi) = f(\pi)$.

$$f_n = \frac{1}{2\pi} \int_{-pi}^{pi} e^{-inx} f(x) \, \mathrm{d}x$$

$$\sum_{n=-\infty\infty} |f_n|^2 = \frac{1}{2\pi} \int_{-pi}^{pi} |f(x)|^2 \, \mathrm{d}x$$

$$S_{n,f} = \sum_{k=-n}^{n} f_k e^{ikx}$$

One should be able to recover f from $S_{n,f}$. In other words, "should be" that

$$\lim_{n\to\infty} S_{n,f}(x) = f(x) \quad \forall x \in [-\pi, \pi]$$

If f is Lipschitz, in the sense taht $\exists\, k > 0$ and $0 < \alpha < 1$ such that for all $x, y \in [-\pi, \pi]$,

$$|f(x) - f(y)| \le k \cdot |x-y|^{\alpha}$$

Then the statement is correct and convergence is uniform.

$$\lim_{n\to\infty} \max_{x\in[-\pi,\pi]} |S_{n,f}(x) - f(x)| = 0$$

This fails in general.

Consequence of B-C-T: $\exists$ a dense G_δ set E in $C_p([-\pi,\pi])$ (where $f \in C_p([-\pi,\pi])$ if $f \in C([-\pi,\pi])$ and $f(-\pi) = f(\pi)$), s.t. $\forall\, f \in E$ the set $\{x \in [-\pi, pi] : \lim_{n\to\infty} S_{n,f}(x) = \infty\}$ is also dense in $[-\pi,\pi]$.

3. Functional analysis: Banach-Steinhauss, Open mapping theorem, Hahn-Banach extension principle.

Theorem 32.5 ***(Banach-Steinhauss)*** *Let X be a Banach space, Y a normed space, and $\{A_\alpha\}_{\alpha\in I}$ a family of continuous linear maps $A_\alpha : X \to Y$. Then either $\exists\, M > 0$ such that $\forall\, \alpha \in I$, $\|A_\alpha\| = \sup_{\|x\|\le 1} \|A_\alpha x\| \le M$ or $\exists$ a dense G_δ set E in X such that $\sup_{\alpha\in I} \|A_\alpha x\| = \infty$ for all $x \in I$.*

Theorem 32.6 ***(Open mapping theorem)*** *Let X and Y be Banach spaces and $A : X \to Y$ a continuous linear onto map. Then for any open set $O \subset X$, the set $A(O) = \{A(x) : x \in O\}$ is open in Y.*

§32.1.1 Banach fixed point theorem (Banach contraction principle)

Definition 32.7 Let X be a metric space. A function $f : X \to X$ is called a contraction if $\exists\, \alpha \in (0,1)$ s.t. $\forall\, x, y \in X$, $d(f(x), f(y)) \le \alpha f(x,y)$.

Theorem 32.8 ***(B-F-P-T)*** *Let X be a complete metric space and $f : X \to X$ a contraction. Then $\exists$ a unique $x \in X$ s.t. $f(x) = x$.*

§33 Friday, November 1, 2013 - Tutorial VII

§33.1 Density of nowhere differentiable functions on $C([0,1])$

Theorem 33.1 *Consider $(C([0,1]), \|\cdot\|_\infty)$ is a complete metric space.*
The set

$$\mathcal{D}^c = \{f \in C([0,1]) : f \text{ is differentiable at some } x \in [0,1]\}^c$$

the set of nowhere differentiable, continuous functions on $[0,1]$, is dense in $C([0,1])$.

Let

$$A_{n,m} = \left\{ f \in C([0,1]) : \exists\, x \in [0,1] \text{ s.t. } \left|\frac{f(t)-f(x)}{t-x}\right| \le n \text{ if } 0 \le |x-t| < \frac{1}{m} \right\}$$

Lemma 33.2 (Lemma 1) If $f \in \mathcal{D} \implies f \in A_{n,m}$ for some n, m.

Proof. If $f \in \mathcal{D}$, say f differentiable at x. We can choose $n \in \mathbb{N}$ such that $|f'(x)| < n$.
$\exists\, \delta > 0$ such that

$$\left|\frac{f(t)-f(x)}{t-x}\right| < n$$

if $|t-x| < \delta$. Pick m such that $\frac{1}{m} < \delta$.
Then, $f \in A_{n,m}$. ∎

So we have $\bigcup_{n,m\in\mathbb{N}} A_{n,m} \supset \mathcal{D}$.

$$\implies \bigcap_{n,m\in\mathbb{N}} A^c_{n,m} = \left(\bigcup_{n,m\in\mathbb{N}} A_{n,m}\right)^c \subset \mathcal{D}^c$$

Lemma 33.3 (Lemma 2) $A^c_{n,m}$ is open ($A_{n,m}$ is closed).

Proof. Let $(f_i)i = 1^\infty$ be a sequence in $A_{n,m}$ and suppose $f_i \to f$ in $(C([0,1]), \|\cdot\|_\infty)$. For each $i \in \mathbb{N}$, $\exists\, x_i \in [0,1]$ such that

$$\left|\frac{f_i(t) - f_i(x_i)}{t - x_i}\right| \le n \qquad \forall\, 0 \le |x_i - t| \le \frac{1}{m}$$

Because $[0,1]$ is compact, $\exists\, \{n_k\}_{k=1}^\infty$ such that $x_{n_k} \to x \in [0,1]$. Then

$$\left|\frac{f(t) - f(x)}{t - x}\right| = \lim_{n_k\to\infty} \left|\frac{f_{n_k}(t) - f_{n_k}(x_{n_k})}{t - x_{n_k}}\right| \le n$$

for all $0 \le |t - x| \le \frac{1}{m}$.

So, $f \in A_{n_m}$. □

Lemma 33.4 (Lemma 3) $PL([0,1])$ the set of all piecewise linear functions is dense in $C([0,1])$.

Proof. Exercise. (use cont. implies unif. cont.) □

Lemma 33.5 (Lemma 4) $A^c_{n,m}$ is dense in $C([0,1])$.

Proof. Let $f \in C([0,1])$, $\varepsilon > 0$ be given.

$\exists\, p \in PL([0,1])$ such that $\|f - p\|_\infty < \frac{\varepsilon}{2}$.

$\exists\, M \in \mathbb{N}$ such that $|p'(x)| \le M$, $\forall\, x \in [0,1]$ (when p is differentiable, which is almost everywhere.

Choose $k > \frac{2(M+n)}{\varepsilon}$.

Choose $\varphi(x) \in PL([0,1])$ with $|\varphi(x)| \le 1$ for all $x \in [0,1]$ and $\varphi'(x) = \pm k$ whenever φ is differentiable at x.

Define $g(x) = p(x) + \frac{\varepsilon}{2}\varphi(x)$.

Then $\|g - p\|_\infty \le \frac{\varepsilon}{2}$. So $\|f - g\| \le \|f - p\| + \|g - p\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$.

Now we want to show $g \in A_{n,m}$. Let $x \in [0,1]$. If g is differentiable at x,

$$|g'(x)| = \left|p'(x) \pm \frac{\varepsilon}{2}k\right|$$

where $k > \frac{2(M+n)}{\varepsilon}$. So we can see

$$|g'(x)| > n$$

If g is not differentiable at x, then x is an endpoint. $\forall x \in [0,1]$, $\exists t$ close enough to x such that

$$\left|\frac{g(t) - g(x)}{t - x}\right| > n$$

So, $g \in A^c_{n,m}$. □

This proves the theorem, by the Baire Category Theorem.

□

§33.2 Banach-Steinhaus theorem

Theorem 33.6 ***(Banach-Steinhaus)*** *Suppose X is a Banach space, Y is a normed space, and $\{\Lambda_\alpha\}_{\alpha \in A}$ is a collection of bounded linear transformations (BLT) from X into Y. Then either $\exists M < \infty$ such that*

$$\|\Lambda_\alpha\| \leq M \qquad \forall \alpha \in A$$

or

$$\sup_{\alpha \in A} \|\Lambda_\alpha x\| = \infty$$

for all $x \in G_\delta$ for G_δ dense in X.

Let $\varphi(x) := \sup_{\alpha \in A} \|\Lambda_\alpha x\|$
Define $V_n := \{x \in X : \varphi(x) > n\}$.

Definition 33.7 $g : X \to \mathbb{R}$ is lower semicontinuous if $\forall\, x \in X$, $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$d(x, y) < \delta \implies g(y) \geq g(x) - \varepsilon$$

Remark. A function $X \to \mathbb{R}$ is continuous iff it is lower-semicontinuous and upper semicontinuous.

Since φ is lower-semicontinous, then

$$V_n = \{x \in X : \varphi(x) > n\} = \varphi^{-1}([(n, \infty))$$

are open $\forall n \in \mathbb{N}$.

Case 1: Suppose V_n for fixed n, is not dense in X. Then $\exists x_0 \in X$, $r > 0$, such that $\overline{B(x_0, r)} \cap V_n = \varnothing$.
If $\|x\| \leq r \implies x_0 + x \notin V_n$.
Then, $\forall \alpha \in A$ and $x \in X$ with $\|x\| \leq r$, $\|\Lambda_\alpha(x_0 + x)\| \leq n$.

$$\|\Lambda_\alpha(x)\| = \|\Lambda_\alpha(x_0 + x - x_0)\| \leq \|\Lambda_\alpha(x_0 + x)\| + \|\Lambda_\alpha(x_0)\| \leq n + n$$

Take $M = \frac{2}{r}$. Then $\forall\, x \in X$ such that $\|x\| \leq 1$, then $\|\Lambda_\alpha(x)\| \leq M$ for all $\alpha \in A$. So $\|\Lambda_\alpha\| \leq M$ for all $\alpha \in A$.

Case 2: Each V_n is dense and open. As X is complete, then

$$G_\delta = \cap_{n=1}^{\infty} V_n$$

is dense.

§33.3 Convergence of Fourier Series

Let $T = [-\pi, \pi]$.

Q. Is it true that $\forall f \in C(T)$, their Fourier series converges to f at each point $x \in T$?
n^{th} partial sum of Fourier series for f at x is given by

$$s_n(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x - t)\, \mathrm{d}t$$

where

$$D_n(t) := \sum_{k=-n}^{n} e^{ikt}$$

(Rudin "Real and Complex" 4.26).
Is $\lim_{n \to \infty} s_n(f, x) = f(x)$ for all $f \in C(T)$ and $x \in T$?

Define: $s^*(f,x) = \sup_n |s_n(f,x)|$. Begin by taking $x = 0$. Define $\Lambda_n(f) = s_n(f,0)$.

$$\|\Lambda_n n\| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,\mathrm{d}t = \|D_n\|_1 < \infty$$

WTS: $\|D_n\|_1 \to \infty$ as $n \to \infty$ which implies $\|\Lambda_n\| \to \infty$.

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

Multiply by $e^{\pm it/2}$.

$$\begin{aligned}
\left(e^{it/2} - e^{-it/2}\right) D_n(t) &= \sum_{k=-n}^{n} \left(e^{i(k+1/2)t} e e^{i(k+1/2)t}\right) \\
&= 2i\sin((n+1/2)t) \\
\implies D_n(t) &= \frac{\sin((n+1/2t))}{\sin(t/2)}
\end{aligned}$$

We know $|\sin(x)| \le |x|$ fora ll $x \in \mathbb{R}$, so

$$\begin{aligned}
\|D_n(t)\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|\,\mathrm{d}t &\ge \frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin((n+1/2)t)}{t/2}\right|\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^{\pi} |\sin((n+1/2)t)t|\frac{\mathrm{d}t}{t} \\
&= \frac{2}{\pi}\int_0^{(n+1/2)\pi} |\sin(t)|\frac{\mathrm{d}t}{t} \\
&\ge \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{\pi}\int_{(k-1)\pi}^{k\pi} |\sin(t)|\,\mathrm{d}t \\
&\le \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \to \infty \text{ as } n \to \infty
\end{aligned}$$

Fix n, put $g(t) = I$ if $D_n(t) \ge 0$ and $g(t) = -1$ if $D_n(t) < 0$. Then $\exists f_j \in C(T)$ such that $-1 \le f_j \le 1$ and $f_j(t) \to g(t)$ for all t.

$$\begin{aligned}
\lim_{j\to\infty} \Lambda_n(f_j) = \lim_{j\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f_j(t)D_n(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)D_n(t)\,\mathrm{d}t \\
\|\Lambda_n\| \to \quad &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,\mathrm{d}t = \|D\|_1
\end{aligned}$$

So, by Banach-Steinhaus, $\|A_n\| \le M$ for all $n \in \mathbb{N}$ (not true) or $\exists$ G_δ, dense in $C(T)$ such that

$$\sup_n \|\Lambda_n f\| = \infty$$

for all $f \in G_\delta$.

So $s^*(f,0) = \infty$ which implies $\lim_{n\to\infty} s_n(f,0)$ does not exist.

So we've shown there is a dense number of functions which are continuous, but whose Fourier series does not exist at 0. (We can shift to get any other point).

§33.4 Complete spaces with no isolated points

Theorem 33.8 *Let X be a complete metric space which has no isolated points. Then, no countable dense set is G_δ.*

Let E be a countable dense set in X, i.e. $E = \bigcup_{k=1}^{\infty}\{x_k\}$. Since X has no isolated points, then $\{x_k\}$ is closed.

Also assume that $E = \bigcap_{n=1}^{\infty} V_n$ where V_n are dense and open.

Define: $W_n := V_n \setminus \bigcup_{k=1}^{n}\{x_k\}$. Then W_n is open and dense.

So, $W = \bigcup_{n\geq 1} W_n = \varnothing$. This is a contradiction, since the empty set can't be dense.

§34 Tuesday, November 5, 2013

1. Midterm tonight
2. Due date for homework 5 is changed to Tues. Nov. 12
3. Tutorial on Nov. 8 is cancelled. Tutorials continue on Nov. 15

§34.1 Banach Contraction Principle or Banach fixed point theorem

Definition 34.1 Let (X, d) be a metric space and $f : X \to X$. f is called a contraction mapping if there exists $\alpha \in (0, 1)$ such that $\forall x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

Any contraction is uniformly continuous.

Theorem 34.2 ***(Banach fixed point)*** *Let X be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point, that is, there exists unique $x \in X$ such that*

$$f(x) = x$$

§34.1.1 Proof

Uniqueness: Let x, y be such that $f(x) = x$ and $f(y) = y$. We need to show that $x = y$.

$$d(x, y) = d(f(x), f(y)) \leq \alpha d(x, y)$$

$$(1-\alpha)d(x, y) \leq 0, \qquad \alpha \in (0, 1) \qquad d(x, y) \geq 0$$

So $d(x, y) = 0$ so $x = y$.

Existence: Pick $x_0 \in X$. Define

$$x_1 = f(x_0) \qquad\qquad x_2 = f(f(x_0))$$

and inductively,

$$x_n = f(x_{n-1}) = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}(x_0)$$

This defines a sequence x_n for $n = 1, 2, \ldots$ in X.

Let now n, m, N be positive integers and suppose that $n, m \geq N$. Also suppose that $n > m$. Then, by triangle inequality,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ d(x_n, x_n - 1) &= d(f(x_{n-2}), f(x_{n-3})) \leq \alpha d(x_{n-1}, x_{n-2}) \\ &= \alpha d(f(x_{n-2}), f(x_{n-3})) \leq \alpha^2 d(x_{n-2}, x_{n-3}) \leq \ldots \leq \alpha^{n-1} d(x_1, x_0) \end{aligned}$$

So,

$$\begin{aligned} d(x_n, x_m) &\le \alpha^{n-1} d(x_1, x_0) + \alpha^{n-2} d(x_1, x_0) + \ldots + \alpha^m d(x_1, x_0) \\ &= \alpha^m d(x_1, x_0)[1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1-m}] \\ &\le \alpha^m d(x_1, x_0) \sum_{k=0}^{\infty} \alpha^k \\ &\le \frac{\alpha^m}{1-\alpha} d(x_1, x_0) \end{aligned}$$

Since $m \ge N$, and $\alpha \in (0,1)$, $\alpha^m \le \alpha^N$, so $d(x_n, x_m) \le \frac{\alpha^N}{1-\alpha} d(x_1, x_0)$.
In the case $m > n$, one proceeds identically, and inequality is obvious if $n = m$. So

$$d(x_n, x_m) \le \frac{\alpha^N}{1-\alpha} d(x_1, x_0)$$

Then $\lim_{N\to\infty} \alpha^N = 0$ ($0 < \alpha < 1$, $\alpha^N = e^{N \log \alpha}$). So given $\varepsilon > 0$, there exists N such that

$$\frac{\alpha^N}{1-\alpha} d(x_1, x_0) < \varepsilon$$

So, for $n, m \ge N$,

$$d(x_m, x_n) \le \frac{\alpha^N}{1-\alpha} d(x_1, x_0) < \varepsilon$$

So x_n is a Cauchy sequence. Since X is complete, $\lim_{n\to\infty} x_n = x$ and exists.
So, we can look at

$$\begin{aligned} x_{n+1} &= f(x_n) \\ \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} f(x_n) \\ \lim_{n\to\infty} x_{n+1} = x \qquad f(x) &= \lim_{n\to\infty} f(x_n) \end{aligned}$$

So,

$$x = f(x)$$

So, x is the fixed point of f.

□

Remark. Rate of convergence by construction:

$$\begin{aligned} d(x_n, x_m) &\le \frac{\alpha^N}{1-\alpha} d(x_1, x_0) \qquad n \ge N \\ \lim_{n\to\infty} d(x_n, x_N) &= d(x, x_N) \\ |d(x_n, x_N) - d(x, x_N)| &\le d(x_n, x) \end{aligned}$$

by triangle. So,

$$d(x, x_N) \le \lim_{n\to\infty} d(x_n, x_N) \le \frac{\alpha^N}{1-\alpha} d(x_1, x_0)$$

Error estimate.

Remark. It is important that X is complete. Take $X = (0, 1]$ with $d(x, y) = |x - y|$. This is not complete as a metric subspace of $\mathbb{R}$.
$f(x) = \frac{x}{2}$, $|f(x) - f(y)| = \frac{1}{2}|x - y|$. So contraction, but for all x, $f(x) \ne x$.

Remark. It is important that f is a contraction. It is not enough that $d(f(x), f(y)) < d(x, y)$ for $x \neq y$. We need some α *strictly* between 0 and 1.

For example, take $X = [0, \infty)$, usual metric, and $f(x) = x + \frac{1}{x+1}$.

$$\begin{aligned} |f(x) - f(y)| &= \left| x + \frac{1}{x+1} - y - \frac{1}{y+1} \right| = \left| x - y + \frac{y-x}{(x+1)(y+1)} \right| \\ &= |x-y| \left| 1 - \frac{1}{(x+1)(y+1)} \right| < |x-y| \end{aligned}$$

If $x \neq y$. However, f does not have a fixed point.

$$f(x) = x \iff x + \frac{1}{x+1} = x \text{ or } \frac{1}{x+1} = 0$$

which is impossible.

Remark. However, if X is compact, and $d(f(x), f(y)) < d(x, y)$, then there exists a unique $x \in X$ such that $f(x) = x$.

Example 34.3 "By compactness, $\exists \alpha \in (0, 1)$ such that $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$." This is *wrong*!

Take $f(x) = \sin(x)$ on $\left[0, \frac{\pi}{2}\right]$.

$$|\sin x - \sin y| = |\cos(t)| \cdot |x - y|$$

for $t \in (x, y)$. In this case, there is no $\alpha \in (0, 1)$ such that

$$|\sin x - \sin y| \leq \alpha |x - y|$$

Another wrong proof Fix k_n with $0 < k_n < 1$, $k_n \to 1$ as $n \to \infty$. Suppose f is not contraction. Then we can find $x_n, y_n \in X$ such that

$$d(x_n, y_n) \geq k_n d(f(x_n), f(y_n))$$

X is compact, so we can find subsequence x_{n_k}, y_{n_k} such that $x_{n_k} \to x$, $y_{n_k} \to y$.

$$d(x_{n_k}, y_{n_k}) \geq k_{n_k} d(f(x_{n_k}), f(y_{n_k}))$$

Take $k \to \infty$, and we get

$$d(x, y) \geq d(f(x), f(y))$$

But $x = y$ case wasn't excluded!

Remark. There are many fixed point theorems.

Brower fixed point theorem:

$$K = \{x \in \mathbb{R}^N, \|x\|_2 \leq 1\}, \text{ closed unit blal in } \mathbb{R}^N$$

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} x_k^2}$$

$f : K \to K$ continuous function. Then there exists $x \in K$ such that $f(x) = x$. It may not be unique.

The motivation is analysis, but the proof uses topology. Very sophisticated.

§34.1.2 Applications: Integral Equations

Set up: $[a, b]$ closed, bounded interval. We are given:

1. Continuous function $K : [a, b] \times [a, b] \to \mathbb{R}$. "integral curve"
2. Continuous function $g : [a, b] \to \mathbb{R}$ " source term"

3. Number $\lambda \neq 0$, "control parameter"

Given $(1), (2), (3)$ we have equation

$$f(x) = \lambda \int_a^b K(x,y) f(y)\,\mathrm{d}y + g(x), \qquad x \in [a,b]$$

for unknown continuous function $f : [a,b] \to \mathbb{R}$. We would like to know when f exists, when it is unqiue, how to compute it, etc.

Similarity to linear algebra: $\mathbb{R}^n$, A-matrix on $\mathbb{R}^n$, $w \in \mathbb{R}^n$ given vector and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. We are looking for $u \in \mathbb{R}^n$ satisfying:

$$u = \lambda A u + w \tag{$\star$}$$

$(\star)$ is equivalent to

$$(I - \lambda A)u = w$$

If $\frac{1}{\lambda}$ is not an eigenvalue of A, then

$$u = (1 - \lambda A)^{-1} w$$

is the unique solution. If $\frac{1}{\lambda}$ is an eigenvalue, solution may not exist, and if it does, it is not unique.

§35 Thursday, November 7, 2013

§35.1 Applications of the Banach Fixed-Point Theorem

Set up: $[a,b]$ closed, bounded interval. We are given:

1. Continuous function $K : [a,b] \times [a,b] \to \mathbb{R}$. "integral kernel"
2. Continuous function $g : [a,b] \to \mathbb{R}$ " source term"
3. Number $\lambda \neq 0$, "control parameter"

Given $(1), (2), (3)$ we have equation

$$f(x) = \lambda \int_a^b K(x,y) f(y)\,\mathrm{d}y + g(x), \qquad x \in [a,b] \tag{$\star$}$$

for unknown continuous function $f : [a,b] \to \mathbb{R}$. We would like to know when f exists, when it is unique, how to compute it, etc.

$$H(x) = \int_a^b K(x,y) f(y)\,\mathrm{d}y$$

and f is continuous, then H is also continuous on $[a,b]$.

$$|H(x) - H(x')| = \left| \int_a^b (K(x,y) - K(x',y)) f(y)\,\mathrm{d}y \right| \leq \int_a^b |K(x,y) - K(x',y)||f(y)|\,\mathrm{d}y$$

$\|f\| = \sup_{a \leq x \leq b} |f(x)|$, so

$$\leq \|f\| \int_a^b |K(x,y) - K(x',y)|\,\mathrm{d}y$$

$K : [a,b] \times [a,b] \to \mathbb{R}$ is continuous, and $[a,b] \times [a,b]$ is compact, so K is uniformly continuous. So given $\varepsilon > 0$, $\exists \delta > 0$ such that if

$$\sqrt{(x-x')^2 + (y-y')^2} < \delta \implies |K(x,y) - K(x',y')| < \frac{\varepsilon}{(b-a)\|f\|}$$

In particular, if $|x - x'| < \delta$, then $|K(x,y) - K(x',y)| < \frac{\varepsilon}{(b-a)\|f\|}$.

Then, if $|x - x'| < \delta$, then

$$|H(x) - H(x')| \le \|f\| \int_a^b |K(x,y) - K(x',y)| \,\mathrm{d}y < \|f\| \frac{\varepsilon(b-a)}{(b-a)\|f\|} = \varepsilon$$

Consider $(C([a,b]), \|\cdot\|)$, Let $A : C([a,b]) \to C([a,b])$, with

$$(Af)(x) = \int_a^b K(x,y) f(y) \,\mathrm{d}y$$

A is linear: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$.

Moreover, if $M = \max_{(x,y)\in[a,b]\times[a,b]} |K(x,y)|$, then

$$|(Af)(x)| = \left| \int_a^b K(x,y) f(y) \,\mathrm{d}y \right| \le \int_a^b |K(x,y)||f(y)| \,\mathrm{d}y \le M(b-a)\|f\|$$

This is true for all x, so

$$\|A(f)\| \le M(b-a)\|f\|$$

If $f, g \in C([a,b])$, then

$$\|A(f) - A(g)\| = \|A(f-g)\| \le M(b-a)\|f - g\|$$

So A is Lipschitz.

Now, we can rewrite our equation $(\star)$ as

$$f = \lambda A(f) + g$$

Remember: $\mathbb{R}^n$, A-matrix, $A : \mathbb{R}^n \to \mathbb{R}^n$ linear, $u = \lambda A(u) + w \iff (I - \lambda A)u = w$.

We will try to apply the Banach fixed point theorem.

Quantum mechanics

$$\frac{\partial \Psi}{\partial t} = iH\Psi$$

where

$$H\Psi = -\frac{\partial}{\partial x}\Psi + V\Psi$$

and we have the boundary conditions $\Psi(a) = \Psi(b) = 0$.

We want to solve $H\Psi = E\Psi$.

This time though, H is an unbounded operator. But "spiritually" it is the same as what we are doing.

Define a map $G : C([a,b]) \to C([a,b])$, by

$$G(f) = \lambda A(f) + g$$

$$G(f_1) - G(f_2) = \lambda A(f_1 - f_2) \implies \|G(f_1) - G(f_2)\| = \|\lambda A(f_1 - f_2)\| = |\lambda M(b-a)| \|f_1 - f_2\|$$

So if $\alpha := |\lambda M(b-a)| < 1$, G is a contraction.

λ is a control parameter, so choose $\lambda < \frac{1}{M(b-a)}$. Then G is a contraction.

Since $(C([a,b]), \|\cdot\|)$ is a Banach space, the equation

$$G(f) = f$$

has a unique solution. This means that our integral equation

$$f(x) = \lambda \int_a^b K(x,y) f(y) \,\mathrm{d}y + g(x)$$
$$f = G(f) = \lambda A(f) + g$$

Remembering the proof of the Banach fixed point theorem, we can construct our unique solution: $G(f_0), G^2(f_0) = G(G(f_0)), G^N(f_0) = \underbrace{G(G(\ldots G(}_{n \text{ times}}f_0)\ldots))$.

If f is the unique solution,

$$\|f - G^N(f_0)\| \leq \frac{\alpha^N}{1-\alpha}\|G(f_0) - f_0\|$$

Example 35.1 $[0,1]$, $K(x,y) = 1$, g — source.

$$f(x) = \lambda \int_0^1 f(y)\,\mathrm{d}y + g(x)$$

Then $\alpha = |\lambda|$. So, if $|\lambda| < 1$, the Banach fixed point theorem works.

Actually, we can solve it directly:

$$\int_0^1 f(x)\,\mathrm{d}x = \lambda \int_0^1 f(y)\,\mathrm{d}y + \int_0^1 g(x)\,\mathrm{d}x$$

So,

$$(1-\lambda)\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x$$

if $\lambda \neq 1$,

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{1-\lambda}\int_0^1 g(x)\,\mathrm{d}x$$

So if f is a solution, then we have

$$f(x) = \underbrace{\frac{\lambda}{1-\lambda}\int_0^1 g(y)\,\mathrm{d}y}_{\text{constant, } C_\lambda(g)} + g(x)$$

Substituting, we see that f is indeed a solution.

Banach fixed point works for $\lambda < 1$. But we know that for any $\lambda \neq 1$, we actually have a unique solution (in this case):

$$f(x) = C_\lambda(g) + g(x), \qquad C_\lambda(g) = \frac{\lambda}{1-\lambda}\int_0^1 g(y)\,\mathrm{d}y$$

When $\lambda = 1$, critical case:

$$f(x) = \int_0^1 f(x)\,\mathrm{d}x + g(x)$$

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f(y)\,\mathrm{d}y + \int_0^1 g(x)\,\mathrm{d}x$$

So unless $\int_0^1 g(x)\,\mathrm{d}x = 0$, the equation does not have a solution. If $\int_0^1 g(x)\,\mathrm{d}x$, then the equation has infinitely many solutions. Any function of the form

$$f(x) = C + g(x)$$

for any arbitrary constant C, is a solution.

To see this, substitute:

$$\int_0^1 f(y)\,\mathrm{d}y + g(x) = \int_0^1 C + g(y)\,\mathrm{d}y + g(x) = C + \underbrace{\int_0^1 g(y)\,\mathrm{d}y}_{=0} + g(x) = f(x)$$

Conclusion: BFP works for $|\lambda| < 1$. Indeed, for any $\lambda \neq 1$, there is a unique solution: $f = C_\lambda(g)+g$, where $C_\lambda(g) = \frac{\lambda}{1-\lambda}\int_0^1 g(y)\,\mathrm{d}y$. If $\lambda = 1$, there is no solution unless $\int_0^1 g(x)\,\mathrm{d}x = 0$; in that case, there are infinitely many solutions (any f of the form $f = c+g$ for arbitrary constant c is a solution).

This is a special case of Fredfoln Theory of compact operators.

Back to our equation in $C([a,b])$:

$$f = \lambda A(f) + g$$

where $A(f) = \int_a^b K(x,y)f(y)\,\mathrm{d}y$.

To be continued.

§36 Friday, November 8, 2013

Last class: $[a,b]$

$$f(x) = \lambda \int_a^b k(x,y)f(y)\,\mathrm{d}y + g(x)$$

$M = \sup|k(x,y)|$. for $|\lambda| < \frac{1}{M(b-a)}$

Banach fixed point $\implies$ existence and uniqueness of solution f.

Example 36.1 $[0,1]$. $k(x,y) = 1$, $f(x) = \lambda\int_0^1 f(y)\,\mathrm{d}y + g(x)$.

Existence and uniqueness fail is $\lambda = 1$.

$$Af(x) = \int_a^b k(x,y)f(y)\,\mathrm{d}y$$

A is linear, $A : C([a,b]) \to C([a,b])$.

$$\|A(f_1) - A(f_2)\| \le M(b-a)\|f_1 - f_2\|$$

A is "very special"

$$B = \{f \in C([a,b]), \|f\| \le 1\}$$

$C([a,b])$ is infinite dimensional; B is closed, bounded, but not compact.

Proposition 36.2

$$\mathrm{cl}(A(B)) = \mathrm{cl}(\{Af : \|f\| \le 1\})$$

is a compact subset of $C([a,b])$.

Proof. By A-A we need to show that $A(B)$ is bounded and equicontinuous. Then $\mathrm{cl}(A(B))$ is compact. If $\|f\| \le 1$, then $\|Af\| \le M(b-a)$. So $A(B)$ is bounded.

To prove equicontinuity,

$$\begin{aligned} |Af(x) - Af(x')| &= \left|\int_a^b [k(x,y)f(y) - k(x',y)f(y)]\,\mathrm{d}y\right| \\ &\le \int_a^b |k(x,y) - k(x',y)| \cdot |f(y)|\,\mathrm{d}y \le \int_a^b |k(x,y) - k(x',y)|\,\mathrm{d}y \end{aligned}$$

By uniform continuity of K, $\exists\,\delta > 0$ s.t. $\forall\, x, x', y \in [a,b]$, if $|x - x'| < \delta$, $|k(x,y) - k(x',y)| < \frac{\varepsilon}{b-a}$. Then for all $f \in B$, if $|x - x'| < \delta$, $|Af(x) - Af(x')| \le \int_a^b \frac{\varepsilon}{b-a}\,\mathrm{d}y = \varepsilon$.

Equicontinuity follows. □

A is an example of a compact operator. A is "infinite dimensional", but "as close as possible" to finite matrices.

Definition 36.3 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. A linear continuous map $A : X \to Y$ is called compact if $\mathrm{cl}\{Af : \|f\|_X \le 1\}$ is a compact subset of Y.

Back to Banach fixed point. X-metric space, $f : X \to X$. f may not be a contraction but a modification of BFP can be used to prove that f has a unique fixed point.

Proposition 36.4 Let X be a metric space and $f : X \to X$ a map s.t. for some k, $f^k = f \circ f \circ \cdots \circ f$ has a unique fixed point. Then f has a unique fixed point.

Proof. Let x be the unique fixed point of f^k.

$$\begin{aligned} f^k(x) &= x \\ f(f^k(x)) &= f^{k+1}(x) = f(x) \\ \implies f^k(f(x)) &= f(x) \end{aligned}$$

Then $f(x)$ is also a fixed point of f^k. So by uniqueness, $f(x) = x$. If $f(y) = y$, then $f(f(y)) = f(y) = y \implies f^k(y) = y \implies x = y$. ☐

Proposition 36.5 Let $(X, \|\cdot\|)$ be a Banach space and $A : X \to X$ a continuous linear map. Suppose that for some k,

$$A^k = A \circ A \circ \cdots \circ A$$

is a contraction. Then for any $y \in X$, $x = Ax + y$ has a unique solution.

Proof. Fix y and let $f : X \to X$ be defined by $f(x) = Ax + y$.

$$\begin{aligned} f^2(x) &= f(f(x)) = f(Ax+y) = A(Ax+y) + y = A^2x + Ay + y \\ f^3(x) &= f(A^2x + Ay + y) = A^3x + A^2y + Ay + y \\ f^k(x) &= A^kx + A^{k-1}y + \cdots + Ay + y \end{aligned}$$

So, $\|f^k(x) - f^k(x')\| = \|A^kx - A^kx'\|$. Since A^k is a contraction, f^k has a unique fixed point. By previous proposition, f has a unique fixed point. ☐

§36.1 Another class of integral equations

$[a,b]$, $k : [a,b] \times [a,b] \to \mathbb{R}$ continuous. $g : [a,b] \to \mathbb{R}$ continuous, $\lambda \in \mathbb{R} - \{0\}$.

$$f(x) = \lambda \int_a^x k(x,y) f(y)\,\mathrm{d}y + g(x)$$

When does this have a solution? When is it unique?

Consider $C([a,b])$.

$$Af(x) = \int_a^x k(x,y) f(x)\,\mathrm{d}y$$

(Exercise: if f is continuous, then Af is also continuous on $[a,b]$). $A : C([a,b]) \to C([a,b])$ is a continuous linear map.

$$\begin{aligned} |Af_1(x) - Af_2(x)| &= \left| \int_a^x k(x,y)(f_1(y) - f_2(y))\,\mathrm{d}y \right| \\ &\le \int_a^x |k(x,y)||f_1(y) - f_2(y)|\,\mathrm{d}y \le \|f_1 - f_2\| M(b-a) \qquad M = \max_{(x,y)} |k(x,y)| \\ \|A(f_1) - A(f_2)\| &\le M(b-a)\|f_1 - f_2\| \text{ if } |\lambda| < \frac{1}{M(b-a)} \end{aligned}$$

BFP $\implies$ existence and uniqueness.

Let us try to iterate A.

$$Af(x) = \int_a^x k(x,y)f(y)\,\mathrm{d}y \qquad A^2 f(x) = \int_a^2 k(x,y)(Af)(y)\,\mathrm{d}y$$

$$A^2 f(x) = \int_a^x k(x,y)\left[\int_a^y k(y,t)f(t)\,\mathrm{d}t\right]\mathrm{d}y = \int_a^x \int_a^y k(x,y)k(y,t)\,\mathrm{d}t\,\mathrm{d}y$$

$$(A^k f)(x) = \int_a^x \int_a^{x+1} \cdots \int_a^{x_{n-1}} k(x,x_1)k(x_1,x_2)\cdots k(x_n,y)f(y)\,\mathrm{d}y\,\mathrm{d}x_{n-1}\cdots\mathrm{d}x$$

$$|A^k f(x)| \le \int_a^x \int_a^{x+1} \cdots \int_a^{x_{n-1}} |k(x,x_1)||k(x_1,x_2)||\cdots|k(x_n,y)||f(y)|\,\mathrm{d}y\,\mathrm{d}x_{n-1}\cdots\mathrm{d}x$$

$$\le \|f\| M^k \underbrace{\int_a^x \int_a^{x+1} \cdots \int_a^{x_{n-1}} \mathrm{d}y\,\mathrm{d}x_{n-1}\cdots\mathrm{d}x}_{\frac{(b-a)^k}{k!}}$$

$$|A^k f(x)| \le \|f\| \frac{M^k(b-a)^k}{k!}$$

$$\|A^k(f_1) - A^k(f_2)\| \le \|f_1 - f_2\| \frac{M^k(b-a)^k}{k!}$$

$e^\alpha = \sum_{k=0}^\infty \frac{\alpha^k}{k!}$, $\lim_{k\to\infty} \frac{\alpha^k}{k!} = 0$ for all α.

$\alpha = M(b-a)$. For large enough k, $\frac{M^k(b-a)^k}{k!} < 1$. So A^k is a contraction. So $f = \lambda Af + g$ has a unique fixed point. $f(x) = \lambda \int_a^x k(x,y)f(y)\,\mathrm{d}y + g(x)$ has a unique solution.

§37 Tuesday, November 12, 2013

§37.1 Topological Spaces

Definition 37.1 Let X be a set and τ a collection of subsets. Then τ is called a topology for X if the following holds

1. $\varnothing, X \in \tau$
2. If $O_1, \ldots, O_n \in \tau$ then $O_1 \cap \cdots \cap O_n \in \tau$
3. If $\{O_i\}_{i\in I}$ is any family of elements of τ then $\bigcup_{i\in I} O_i \in \tau$

The elements of τ are called open sets. If $x \in X$, any open set that contains x is a neighborhood of x.

Basic example: Let (X,d) be a metric space and τ the collection of all open sets in X. Then τ is a topology for X.

Very important example: distribution, modern theory of partial differential equations

δ-function,

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

$\delta : C(\mathbb{R}) \to \mathbb{R}$, $\delta(f) = f(0)$ if differentiable, $\delta'(f) = f'(0)$. Ex. number theory: Zarisky topology.

§37.2 Basic notions

Definition 37.2 Let (X,τ) be a topological space and a collection of neighborhoods β_x of x is called base for topology at x if for any neighborhood O of x, one can find $B \in \beta_x$ such that $B \subset O$.

A family of open sets in β in X is called a base for topology τ if β contains a base for a topology for any point $x \in X$. In other words, β is a base for the topology τ if for any $x \in X$, and any neighborhood O at x, one can find $B \in \beta$ such that $x \in B \subset O$.

Proposition 37.3 Let β be a collection of open sets in topological space (X, τ). T.F.A.E.

1. β is a base for a topology τ
2. Any non-empty open set in τ can be written as a union of some subcollection of β

Proof. (1) $\implies$ (2): Let O be a non-empty open set. Let $x \in O$. Then O is a neighborhood of x, and β is a base for topology at x. So there exists $B_x \in \beta$ such that $x \in B_x \subset O$

$$\bigcup_{x \in O} B_x = O$$

so (2) follows.

(2) $\implies$ (1): Let $x \in X$ be given. We need to show β contains a base for topology at x. Hence, we need to show that if O is a non-empty open set containing x, then there is $B \in \beta$ such that

$$x \in B \subset O$$

O is open, hence $O = \bigcup_{i \in I} B_i$ for some family $\{B_i\}_{i \in I}$ in β. But then $x \in O \implies x \in B_{i_0}$ for some $i_0 \in I$. So $x \in B_{i_0} \subset O$ and (1) follows. □

A consequence: Base uniquely determines a topology. In other words, if τ_1 and τ_2 are two topologies with the same base β, then $\tau_1 = \tau_2$. This follows from the above proposition: If $O \in \tau_1$, then $O = \bigcup_{i \in I} B_{i_0}$ where $B_i \in \beta$. $B_i \in \tau_2$, so $O \in \tau_2$. Can do this the other way, so $\tau_1 = \tau_2$.

However, given topology can have many different bases.

Example 37.4 $\mathbb{R}$, $d(x, y) = |x - y|$, metric topology. One base is a collection of all the open intervals whose ends are rational points. Another base is a collection of all open intervals whose ends are irrational points.

Proposition 37.5 Let X be a set and β a collection of subsets of X. T.F.A.E.

1. β is a base for a topology for X
2. β covers X, that is, $\bigcup_{B \in \beta} B = X$, and for any $B_1, B_2 \in \beta$ and any $x \in B_1 \cap B_2$, there exists $B \in \beta$ such that $x \in B \subset B_1 \cap B_2$.

Proof. (1) $\implies$ (2): β is a base for some topology τ in X. If $x \in X$, X is open, so there is $B_x \in \beta$ such that $x \in B_x \subset X$. Then

$$\bigcup_{x \in X} B_x = X \text{ and so } \bigcup_{B \in \beta} \bigcup B = X$$

If $B_1, B_2 \in \beta$ and $x \in B_1 \cap B_2$, then $B_1 \cap B_2$ is neighborhood of x. Since β is a base at x, there is $B_x \in \beta$ such that $x \in B_x \subset B_1 \cap B_2$.

Let τ be the collection of all unions of subcollections of β together with $\varnothing$. $A \in \tau$ if either $A = \varnothing$ or $A = \bigcup_{i \in I} B_i$ where $\{B_i\}$ is a collection of sets in β.

We need to show that τ is a topology for x. $\varnothing \in \tau$ and since β covers X, $X \in \tau$. We need to show that τ is closed under arbitrary unions. Let $\{A_j\}_{j \in J}$ be a collection of sets in τ indexed by the set J.

Then each A_j is a union of some subcollection of β. We can write

$$A_j = \bigcup_{i \in I_j} B_{(i,j)}, \qquad B_{(i,j)} \in \beta$$

$$\bigcup_{j \in J} A_j = \bigcup_{j \in J} \bigcup_{i \in I_j} B_{(i,j)}$$

union and a family of sets in β.

By definition, τ is closed under arbitrary union. We need to show that τ is closed under finite intersection. Enough to show that if $A_1, A_2 \in \tau$¡ then $A_1 \cap A_2$ is also in τ.

Let $A_1 = \bigcup_{i \in I} B_{ij} \qquad A_2 = \bigcup_{j \in J} B_j, \qquad B_i, B_j \in \beta.$

$$A_1 \cap A_2 = \bigcup_{i \in I, j \in J} B_1 \cap B_j$$

Let $x \in A_1 \cap A_2$. Then there exists i_0, j_0 such that $x \in B_{i_0} \cap B_{j_0}$. Then, by (2), there exists $B_x \in \beta$ such that $x \in B_x \subset B_{i_0} \cap B_{j_0}$.

$$\bigcup_{x \in A_1 \cap A_2} B_x \supset A_1 \cap A_2 \qquad \text{since } x \in B_x$$

$$\bigcup_{x \in A_1 \cap A_2} B_x \subset A_1 \cap A_2 \qquad \text{since } B_x \subset B_{i_0} \cap B_{j_0} \subset A_1 \cap A_2$$

So,

$$\bigcup_{x \in A_1 \cap A_2} B_x = A_1 \cap A_2$$

and so, $A_1 \cap A_2$ is also a union of some subcollection of the elements of β. So τ is closed under finite intersections. $\square$

Definition 37.6 Let (X, τ) be a topological space and $E \subset X$. A point $x \in X$ is called a closure point of E if for any neighborhood O_x of x, $O_x \cap E \neq \varnothing$.

$\operatorname{cl} E$ is the collection of all closure points.

Obviously, $E \subset \operatorname{cl} E$, and if $A \subset B \implies \operatorname{cl} A \subset \operatorname{cl} B$.

Proposition 37.7 $\operatorname{cl}(\operatorname{cl} E) = \operatorname{cl} E$

§38 Thursday, November 14, 2013

Last class: (X, τ) topological space, $E \subset X$. A point $x \in X$ is called a closure point of E if for any neighborhood O_x of x, $O_x \cap E \neq \varnothing$.

$\operatorname{cl} E$ or $\bar{E}$ is the collection of all closure points of E. $\operatorname{cl} E$ is called the closure of E. Obviously, $E \subset \operatorname{cl} E$ and $A \subset B \implies \operatorname{cl}(A) \subset \operatorname{cl}(B)$.

A set E is called closed if $\operatorname{cl} E = E$.

Proposition 38.1 $\operatorname{cl} E$ is a closed set.

Proof. We need to show that $\operatorname{cl}(\operatorname{cl}(E)) = \operatorname{cl}(E)$. Since $\operatorname{cl}(E) \subset \operatorname{cl}(\operatorname{cl}(E))$, we need to show that $\operatorname{cl}(\operatorname{cl}(E)) \subset \operatorname{cl} E$.

Let $x \in \operatorname{cl}(\operatorname{cl}(E))$. Let O_x be any neighborhood of x. Then $O_x \cap \operatorname{cl} E \neq \varnothing$. So, $\exists x' \in O_x \cap \operatorname{cl} E$. So x' is a closure point of E, and O_x is an open set that contains x'. So by the definition of a closure point, applied to x', we have

$$O_x \cap E \neq \varnothing$$

so $x \in \operatorname{cl} E$, and we have that $\operatorname{cl} E$ is closed. $\square$

Remark. Notice that $\operatorname{cl} E$ is the smallest closed set that contains E. To see this, let F be a closed set with $E \subset F$. Take $x \in F^c$. Since $\operatorname{cl} F = F$, x is not a closure point of F. So, $\exists\, O_x \ni x$ open with $O_x \cap F = \varnothing$.

But $E \subset F$, so $O_x \cap E = \varnothing$. Then $x \notin \operatorname{cl} E$. So we have $x \in F^c \implies x \in (\operatorname{cl}(E))^c$. Taking a closure,

$$x \in F \implies x \in \operatorname{cl} E$$

i.e. $\operatorname{cl} E \subset F$.

Proposition 38.2 Let (X, τ) be a topological space. A set $E \subset X$ is open if and only if E^c is closed.

Proof. Suppose that E is open. Let $x \in \operatorname{cl}(E^c)$. Then $x \notin E$, since E is open (since it would be a neighborhood of x), and $E \cap E^c = \varnothing$. Then $x \in \operatorname{cl}(E^c) \implies x \in E^c$ and $\operatorname{cl}(E^c) \subset E^c$, so E^c is closed.

To prove the opposite inclusion, suppose that E^c is closed. Let $x \in E$. Then $x \notin \operatorname{cl}(E^c)$, since $\operatorname{cl}(E^c) = E^c$. Then we can find a neighborhood O_x that contains x and is completely contained in E.

So,

$$E = \bigcup_{x \in E} O_x$$

where E is a union of open sets, and hence open. □

Proposition 38.3 Let (X, τ) be a topological space. Then

1. $\varnothing, X$ are closed
2. If $F_1, \ldots, F_n$ are closed, $F_1 \cup \cdots \cup F_n$ is a closed set
3. If $\{F_i\}_{i \in I}$ is a family of closed sets, then $\bigcup_{i \in I} F_i$ is closed.

§38.0.1 Four Separation properties

(X, τ).

1. Tychonoff separation property: if $x, y \in X$, $x \neq y$, then there exists a neighborhood O_x of x such that $y \notin O_x$ and a neighborhood O_y of y such that $x \notin O_y$.
2. Hausdorff separation property: If $x, y \in X$, $x \neq y$, then there are neighborhoods O_x and O_y that obey
$$O_x \cap O_y = \varnothing$$
3. Regular separation property: Tychonoff property holds, and, if F is a closed subset of X, and $\exists\, x \notin F$, then $\exists$ an open set $V \supset F$ and a neighborhood O_x at x such that
$$O_x \cap V = \varnothing$$
4. Normal separation: Tychonoff property holds, and, for any two disjoint closed sets F_1, F_2, there are open sets O_1 and O_2 such that $F_1 \subset O_1$, $F_2 \subset O_2$, and $O_1 \cap O_2 = \varnothing$.

Proposition 38.4 Tychonoff's property holds if and only if any single point in X is a closed set.

Proof. Suppose that T.P. holds. Let $x \in X$, $y \neq x$. Then there is a neighborhood of O_y of y s.t. $x \notin O_y$. So,

$$\{x\}^c = \bigcup_{y \in X \setminus \{x\}} O_y$$

this is a union of open sets and hence open.

To prove the converse, suppose that any point x is a closed set. Fix x and let $y \neq x$. Then $y \in \{x\}^c$ which is open. So $\{x\}^c$ is a neighborhood of y that does not contain x. Reversing the roles of x and y, we have the same statement. So T.P. holds. □

Proposition 38.5 Any metric space is normal.

Proof. (X, d) metric space, $F \subset X$, $\varnothing \neq F$.
Homework 2:

$$d(x, F) = \inf\{d(x, y) : y \in F\}$$

If F is closed, $d(x, F) = 0 \iff x \in F$.

Homework 6: Function $x \to d(x, F)$ is uniformly continuous, in fact

$$|d(x, F) - d(y, F)| \leq d(x, y)$$

Let F_1, F_2 be two closed sets in X. We want to find open sets $O_1 \supset F_1$, $O_2 \supset F_2$, s.t. $O_1 \cap O_2 = \varnothing$. If F_1 or F_2 is empty, this is trivial.

Assume that F_1, F_2 are non-empty.

Let

$$\begin{aligned}
f(x) &= d(x, F_1) - d(x, F_2) \\
g(x) &= d(x, F_2) - d(x, F_1) \\
O_1 &= \{x \in X : f(x) < 0\} = f^{-1}((-\infty, 0)) \\
O_2 &= \{x \in X : g(x) < 0\} = g^{-1}((-\infty, 0))
\end{aligned}$$

O_1, O_2 are open since they are continuous.

Finally, $F_1 \subset O_1$, since for $x \in F_1$, $d(x, F_1) = 0$, and $d(x, F_2) > 0$ since F_2 is closed and $F_1 \cap F_2 = \varnothing$. In the same way, $F_2 \subset O_2$.

So we have found appropriate O_1, O_2.

□

Topological space is called Tychonoff/Hausdorff/regular/normal if the respective separation property holds.

Metric $\implies$ normal $\implies$ regular $\implies$ Hausdorff $\implies$ Tychonoff.

Exercise 38.6 Let $X = \mathbb{R}$ and $\tau = \{\varnothing, \mathbb{R}, (-\infty, c) : c \in \mathbb{R}\}$.

Then τ is a topology for $\mathbb{R}$.

This topology is Tychonoff, but not Hausdorff.

Update: This topology is not Tychonoff. See the next day's notes.

§38.1 Convergence

Definition 38.7 Convergence: let (X, τ) be a topological space and $(x_n)_{n=1}^{\infty}$ a sequence in X This sequence converges to a point $x \in X$, denoted $\lim_{n\to\infty} x_n = x$, if for any neighborhood O_x of x, there is $N > 0$ such that for all $n \geq N$, $x_n \in O_x$.

Example 38.8 ***(Example 1, continuation)*** $x_n = -n$. Then for any $x \in \mathbb{R}$, $\lim_{n\to\infty} x_n = x$. Let O_x be a neighborhood of x. So $O_x = \mathbb{R}$, or $O_x = (-\infty, c)$ with $c > x$. Then, for n large enough, $x_n = -n < c$, so $x_n \in O_x$. Then $x_n \to x$.

§39 Friday, November 15, 2013

Example 39.1 ***(Example 1, correction)*** $X = \mathbb{R}$, $\tau = \{\varnothing, \mathbb{R}, (-\infty, c) : c > 0\}$. τ is a topology for $\mathbb{R}$. τ is *not* Tychonoff.

Proof. To prove τ is a topology, we need to show τ is closed under finite intersection and arbitrary union.

If $O_1, \ldots, O_n \in \tau$, we consider $O_1 \cap \ldots \cap O_n$. If $O_k = \mathbb{R}$, we can drop it. If $O_k = \varnothing$, the statement is trivial.

So, WLOG, $O_k \neq \varnothing$, $O_k \neq \mathbb{R}$, for $1 \leq k \leq n$.

Then $O_k = (-\infty, c_k)$. $\bigcap_{k=1}^{\infty} O_k = (-\infty, c)$, $c = \min_{1\leq k\leq n} c_k$. So τ is closed under finite intersections.

Let $\{O_i\} = (-\infty, c_i)$ for $c_i \in \mathbb{R}$.

Let $c = \sup_{i\in I} c_i$, then $(-\infty, c_i) \subset (-\infty, c)$, so $\bigcup_{i\in I} O_i \subset (-\infty, c)$. If $c < \infty$, then $c - 1/n$ is not an upper bound for $\{c_i\}_{i\in I}$. So, $(-\infty, c - 1/n) \subset O_{i_n} \subset \bigcup_{i\in I} O_i$. Then $\bigcup_{n=1}^{\infty}(-\infty, c - 1/n) \subset \bigcup_{i\in I} O_i$. So, if $c < \infty$ we have equality.

$$\bigcup_{i\in I} O_i = (-\infty, c)$$

If $c = \infty$, then for any n, $\exists\, c_{in}$ s.t. $c_{in} > n$. So, $(-\infty, n) \subset \bigcup_{i\in I} O_i$ for all n.

So $(-\infty, n) \subset \bigcup_{i\in I} O_i$ for all n, and so $(-\infty, \infty) \subset \bigcup_{i\in I} O_i$.

τ is a topology. τ is not Tychonoff. Let $x, y \in \mathbb{R}$ with $x < y$. Then any open set that contains y must be of the form $(-\infty, c)$ for some $c > y$. But then $x \in (-\infty, c)$ so we can't separate x from y. Thus, X is not Tychonoff.

$x_n = -n$. Then for any $x \in \mathbb{R}$, $\lim_{n\to\infty} x_n = x$. Let O_x be a neighborhood of x. So $O_x = \mathbb{R}$, or $O_x = (-\infty, c)$ with $c > x$. Then, for n large enough, $x_n = -n < c$, so $x_n \in O_x$. Then $x_n \to x$. □

Example 39.2 *(Example 2)* $X = \mathbb{R}$, $\tau = \{\varnothing, X, O : O^c \text{ is finite set}\}$.

τ is a topology: if $O_1, \ldots, O_n \in \tau$, $O_k \neq \varnothing$ s.t. $O_k \neq \mathbb{R}$, then

$$(O_1 \cap \cdots \cap O_n)^c = O_1^c \cup \cdots O_n^c$$

which is a finite set, since each O_k^c is finite.

If $\{O_i\}_{i\in I}$ is a family in τ, with $O_i \neq \varnothing$, $O_i \neq \mathbb{R}$, then

$$\left(\bigcup_{i\in I} O_i\right)^c = \bigcap_{i\in I} O_i^c$$

which is a finite set since each O_i^c is finite.

So, τ is closed under arbitrary union and finite intersection.

Let $x, y \in \mathbb{R}$, $x \neq y$. Let $O_y = \mathbb{R} \setminus \{x\}$ and $O_x = \mathbb{R} \setminus \{y\}$. Then O_x neighborhood of x, $y \notin O_x$. O_y is a neighborhood of y, $x \notin O_y$.

Tychonoff holds, Hausdorff however does not. Suppose that V_x and V_y are neighborhoods of x and Y such that $V_x \cap V_y = \varnothing$, V_x^c, V_y^c is finite. But $V_x^c \cup V_y^c = \mathbb{R}$ which is a contradiction.

Let $x_n = n$, then for any $x \in \mathbb{R}$, $\lim_{n\to\infty} x_n = x$. To see this, let O be a neighborhood of the point $x \in \mathbb{R}$ with $O \neq \mathbb{R}$. Then O^c consists of finitely many points. Sequence x_n consists of distinct points, so $\exists\, N > 0$ such that $x_n \notin O^c$ for $n \geq N$. But then $x_n \in O$ for $n \geq N$, and $\lim_{n\to\infty} x_n = x$.

Proposition 39.3 Let (x, τ) be a Hausdorff space. Then the limit point of a sequence, if it exists, is unique.

Proof. Let $(x_n)_{n=1}^\infty$ be a sequence such that

$$\lim_{n\to\infty} x_n = x$$

and

$$\lim_{n\to\infty} x_n = y$$

with $x \neq y$. Let O_x and O_y be neighborhoods of x and y such that $O_x \cap O_y = \varnothing$. Let N_1 be such that for $n \geq N1$, $x_n \in O_x$. Let N_2 be such that for $n \geq N_2$, $x_n \in O_y$. Then, for $n \geq \max(N_1, N_2)$, $x_n \in O_x \cap O_y$. Contradiction. □

Definition 39.4 (X, τ) a topological space. X is called first countable if it has a countable base for topology τ at each point $x \in X$.

X is called second countable if it has a countable base for topology τ.

Remark. Second countable $\implies$ first countable.

Remark. A metric space is first countable. Let (X, d) be a metric space, $x \in X$, $\{B(x, {}^1\!/\!_n)\}_{n>0}$ is a countable base for a topology at x.

Definition 39.5 Let (X, τ) be a topological space. A set $\mathcal{S} \subset X$ is called dense in X if every non-empty open set in X contains an element of $\mathcal{S}$.

If X contains a countable dense set, X is called separable.

Remark. Note that $\mathcal{S}$ dense in X if and only if $\operatorname{cl}\mathcal{S} = X$. Suppose $\operatorname{cl}\mathcal{S} = X$ and O a non-empty open set. Pick $x \in O$. Then O is a neighborhood of x. $x \in \operatorname{cl}\mathcal{S} = X$, so $\mathcal{S} \cap O \neq \varnothing$. Conversely, suppose that $\mathcal{S}$ is dense in X. Then for $x \in X$ and any neighborhood O of x, then $O \cap \mathcal{S} \neq \varnothing$, so $x \in \operatorname{cl}\mathcal{S}$.

Remark. If topological space is second countable, then it is separable. Indeed, let $\{B_n\}_{n=1}^\infty$ be a countable base for topology, and $\mathcal{S} = \{x_n\}_{n=1}^\infty$. Pick $x_n \in B_n$; any open set O in X can be written as a union of some subcollection of $\{B_n\}$. Hence, any non-empty open set contains a point of $\mathcal{S}$.

However, if topological space X is separable, it may not be second countable. However, a metric space is separable iff it is second countable.

Recall: if $\{x_n\}_{n=1}^\infty$ is a dense set in metric space (X, d), we have shown that $\{B(x_n, {}^1\!/\!_m\}_{n,m\geq 0}$ is a base for topology for X. Any open set in a metric space can be written as a union of these balls.

Definition 39.6 For a topological space (X, τ), we say (X, τ) is metrizable if there exists a metric d such that the topology of open sets with respect to this metric is precisely τ.

Example 39.7 Let X be a non-empty set and $\tau = \mathcal{P}(X) = 2^X =$ the set of all subsets of X. (τ is the maximal topology).

τ is metrizable: just use the discrete metric. Any point is an open set ($B(x, 1/2) = \{x\}$), and any set is open.

Example 39.8 Let X be a set with more than two points and let $\tau_{\min} = \{\varnothing, X\}$. It is not metrizable. If d exists, take $x \neq y$ with $x, y \in X$, and $r = d(x, y) > 0$. Then $B(x, r) = \{z : d(x, z) < r\}$. $x \in B(X, r)$, but $y \notin B(x, r)$. So this ball is not the empty set or X, so d cannot exist.

§40 Friday, November 15, 2013: Tutorial

§40.1 Picard's local existence theorem

Theorem 34.2 ***(Banach fixed point)*** *Let X be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point, that is, there exists unique $x \in X$ such that*

$$f(x) = x$$

If T is a contraction, $\exists\, q \in [0, 1)$ s.t. $d(T(x), T(y)) \leq q d(x, y)$.

Consider $d(T(X), T(Y)) < d(x, y)$; if X is compact, we have a unique fixed point (from the homework).

Let $O \subset \mathbb{R}^2$ be open. $(x_0, y_0) \in O$. $g : O \to \mathbb{R}$ continuous.

The problem: find a solution to the following differential equation:

$$f : I \to \mathbb{R} \text{ satisfies } f'(x) = g(x, f(x)) \text{ and } f(x_0) = y_0 \tag{1}$$

$$f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, \mathrm{d}t$$

for $x \in I$. The question: when is this expression valid?

Theorem 40.1 ***(Picard's local existence theorem)*** *Let $(x_0, y_0) \in O \subset \mathbb{R}^2$ be open. Suppose $g : O \to \mathbb{R}$ be continuous and satisfies the following Lipschitz condition: $\exists\, M > 0$ such that*

$$|g(x, y_1) - g(x, y_2)| < M|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in O$.

Then, $\exists$ open interval I such that (1) has a unique solution.

Proof. For $\ell > 0$, set $I_\ell := [x_0 - \ell, x_0 + \ell]$. Then it's enough to show that we can choose ℓ such that $\exists!\, f : I_\ell \to \mathbb{R}$ such that $f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, \mathrm{d}t$.

Since O is open, $\exists\, a, b > 0$ such that $R_{ab} = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subset O$.

For $\ell \leq a$, define

$$X_\ell := \{f \in C(I_\ell) : |f(x) - y_0| \leq b,\ \forall\, x \in I_\ell\}$$

Exercise: X_ℓ is closed.

Define: $T : X_\ell \to C(I_\ell)$, $T(f)(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, \mathrm{d}t$.

Remains to show: ℓ can be chosen such that $T(X_\ell) \subset X_\ell$ and $T : X_\ell \to X_\ell$ is a contraction.

First, R_{ab} is compact, g continuous, so $\exists\, K > 0$ such that $|g(x, y)| \leq K$ for $(x, y) \in R_{ab}$. Now take $f \in X_\ell$ and $x \in I_\ell$. Then

$$|T(f)(x) - y_0| = \left| \int_{x_0}^{x} g(t, f(t))\, \mathrm{d}t \right| \leq \left| \int_{x_0}^{x} K\, \mathrm{d}t \right| = \ell K \leq b \iff \ell \leq \frac{b}{k}$$

Hence, $T(f) \in X_\ell \implies T : X_\ell \to X_\ell$.

Let $f_1, f_2 \in X_\ell$, $x \in I_\ell$.

$$
\begin{aligned}
|g(x, f_1(x)) - g(x, f_2(x))| &\leq M|f_1(x) - f_2(x)| \leq M\|f_1 - f_2\| \\
\implies |T(f_1)(x) - T(f_2)(x)| &= \left| \int_{x_0}^{x} [g(t, f_1(t)) - g(t, f_2(t))] \, \mathrm{d}t \right| \\
&\leq \left| \int_{x_0}^{x} M\|f_1 - f_2\| \, \mathrm{d}t \right| \leq \ell M\|f_1 - f_2\|
\end{aligned}
$$

$$\implies \|T(f_1) - T(f_2)\| \leq \ell M\|f_1 - f_2\|$$

If $\ell \leq \frac{1}{2M}$, then T is a contraction. □

Example 40.2 Suppose $g(x, y) = h(x) + by$ for $h : \mathbb{R} \to \mathbb{R}$ continuous.
Verify: $f(x) = e^{b(x-x_0}y_0 + \int_{x_0}^{x} e^{b(x-t)}h(t) \, \mathrm{d}t$ is a solution to $f'(x) = g(x, f(x))$ and $f(x) = y_0$.

$$
\begin{aligned}
\tfrac{\mathrm{d}f(x)}{\mathrm{d}x} &= be^{b(x-y_0)}y_0 + \tfrac{\mathrm{d}}{\mathrm{d}x}\left[e^{bx}\int_{x_0}^{x} e^{-bt}h(t) \, \mathrm{d}t\right] \\
&= be^{b(x-y_0)}y_0 + \left[be^{bx}\int_{x_0}^{x} e^{-bt}h(t) \, \mathrm{d}t\right] + \left[e^{bx}\tfrac{\mathrm{d}}{\mathrm{d}x}\int_{x_0}^{x} e^{-bt}h(t) \, \mathrm{d}t\right] \\
&= be^{b(x-y_0)}y_0 + \left[be^{bx}\int_{x_0}^{x} e^{-bt}h(t) \, \mathrm{d}t\right] + \left[e^{bx}[e^{-bx}h(x)]\right] \\
&= h(x) + be^{b(x-x_0)}y_0 + \int_{x_0}^{x} e^{b(x-t)}h(t) \, \mathrm{d}t = g(x, f(x))
\end{aligned}
$$

Lemma 40.3 *Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and has continuous first order partial derivatives. Then for each $(x_0, y_0) \in \mathbb{R}^2$, $\exists O \in (x_0, y_0)$ (O open) such that $g : O \to \mathbb{R}^2$ satisfies Picard Thm condition (ie. the uniform Lipschitz condition).*

Proof. Choose $(x_0, y_0) \in \mathbb{R}^2$. $\frac{\partial g}{\partial y}(x, y) = g_y(x, y)$ is continuous, so $\exists\, \delta > 0$ such that

$$|g_y(x, y) - g_y(x_0, y_0)| < 1 \quad \forall (x, y) \in B((x_0, y_0), \delta) =: O$$

Let $M = |g_y(x_0, y_0)| + 1$. Then $|g_y(x, y)| \leq M$ for $x, y \in O$.

Pick $(x_1, y_1), (x_2, y_2) \in O$. Then $|g(x, y_1) - g(x, y_2)| = |g_y(x, y')(y_1 - y_2)| = |g_y(x, y')||y_1 - y_2| \leq M|y_1 - y_2|$, by mean value theorem. □

Example 40.4 $f(x) = x - \arctan(x)$. WLOG, take $x > y$:

$$|f(x) - f(y)| = |x - \arctan x - y + \arctan y| = |\underbrace{(x - y)}_{>0} - \underbrace{(\arctan(x) - \arctan(y))}_{>0}| < |x - y|$$

since $\arctan(x) - \arctan(y) < x - y$. Not a contraction, but has a fixed point on compact X (when $f : X \to X$).

$f : \mathbb{R} \to \mathbb{R}$. For $A > 0$, $f : [-A, A] \to [-A, A]$. Fixed point is 0.

Let $g(x) = \frac{\pi}{2} + x - \arctan(x)$. Suppose g has a fixed point; then $x_0 = \frac{\pi}{2} + x_0 - \arctan(x_0)$ which is impossible.

§41 Tuesday, November 19, 2013

§41.1 Characterization of Closure

In a metric space (X, d), $x \in \operatorname{cl} E$ iff $\exists$ a sequence $(x_n)_{n=1}^{\infty}$ in E s.t. $\lim_{n\to\infty} x_n = x$.

In a topological space, if x is s.t. for some sequence $x_n \in E$, $\lim_{n\to\infty} x_n = x$, then $x \in \operatorname{cl} E$. However, in general, the converse is not true. But if X is first countable, then the converse holds.

Proposition 41.1 Let (X, τ) be a first countable topological space. Let $E \subset X$. Then $x \in \operatorname{cl} E$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in E s.t.

$$\lim_{n\to\infty} x_n = x$$

§41.2 Continuous Functions on Topological Spaces

Simple fact Let (X, τ) be a topological space. Then $V \subset X$ is an open set iff for any $x \in V$, there is a neighborhood V_x at x with $V_x \subset V$.

Proof. If V is open, then take $V_x = V$. To prove the opposite direction, note that

$$V = \bigcup_{x\in V} V_x$$

Hence, V is open. □

Definition 41.2 Let (X, τ) and (Y, s) be two topological spaces. Let $f : X \to Y$ be a function. This function is continuous at the point $x \in X$ if for any neighborhood V at $f(x)$, there exists a neighborhood U at x s.t. $f(U) \subset V$.

If f is continuous at every point of X, f is called a continuous function.

Proposition 41.3 A function $f : X \to Y$ is continuous if and only if the inverse image of any open set in Y is an open set in X.

Proof. Suppose that f is continuous. Let O be an open set in Y. Consider

$$f^{-1}(O) = \{x \in X : f(x) \in O\}$$

Let $x \in f^{-1}(O)$. Then $f(x) \in O$ and O is a neighborhood of $f(x)$. Since f is continuous at x, there is a neighborhood U_x of x in X, s.t. $f(U_x) \subset O$. But then,

$$U_x \subset f^{-1}(O)$$

Hence, for any $x \in f^{-1}(O)$, there is a neighborhood U_x of x s.t. $U_x \subset f^{-1}(O)$. Hence, $f^{-1}(O)$ is open.

To prove the converse, suppose that $f^{-1}(O)$ is open in X, for any open set $O \subset Y$.

Let $x \in X$ be given, and let V be a neighborhood of $f(x)$. Then V is open and contains $f(x)$, and so $f^{-1}(V)$ is open in X (by assumption) and $x \in f \in v(V)$. Set $U = f^{-1}(V)$. Then U is a neighborhood of x and $f(U) = f(f^{-1}(V)) \subset V$. So f is continuous at x. □

Proposition 41.4 Let $(X, \tau), (Y, s), (Z, \nabla)$ be topological spaces and

$$f : X \to Y \qquad g : Y \to Z$$

be continuous functions. Then their composition

$$h = g \circ f : X \to Z$$

is continuous.

Proof. This is true since for any open set $O \subset Z$, $h^{-1}(O) = (g \circ f)^{-1}(O) = \underbrace{f^{-1}(\underbrace{g^{-1}(O)}_{\text{open in } Y})}_{\text{open in } X}$. □

Remark. One can use the requirement of continuity to generate a topology. This is very useful.

Let X be a set. Let A be a collection of subsets of X. We do not know whether A is a topology (typically, it is not). We want to find the smallest topology for X that contains A.

Let τ_A be the collection of all topologies for X that contain A. τ_A is not empty: power set of X is in τ_A. Define now,

$$\tau = \bigcap_{S \in \tau_A} S$$

τ is the miminal topology that contains A, as can be quickly checked.

τ is the topology generated by A.

X-set. $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$ a family of topological spaces. For each α, we are given a function $f_\alpha : X \to X_\alpha$. For the f_α to be continuous, then we need

$$A = \{f^{-1}(O) : O \in \tau_\alpha, \alpha \in I\}$$

to contain only open sets. The topology genenerated by A is the minimal topology in X with respect to which all functions f_α are continuous.

§41.3 Compact Topological Spaces

Definition 41.5 Let (X, τ) be a top. space. X is called compact if every open cover of X has a finite subcover. That is, if

$$X = \bigcup_{\alpha \in I} V_\alpha$$

where $V_\alpha \in \tau$, then there are finitely many $V_{\alpha_1}, \ldots, V_{\alpha_n}$ s.t. $X = V_{\alpha_1} \cup \ldots V_{\alpha_n}$.

Definition 41.6 (X, τ) has the finite intersection property if for any family of closed set $\{F_\alpha\}_{\alpha \in I}$ such that $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} \neq \varnothing$ for any $\alpha_1, \ldots, \alpha_n \in I$, we have that

$$\bigcap_{\alpha \in I} F_\alpha \neq \varnothing$$

Proposition 41.7 (X, τ) is compact iff (X, τ) has the finite intersection property.

Proof. Same as for metric spaces. □

Definition 41.8 Let (X, τ) be a top. space. A subset $Y \subset X$ is called compact if any open cover of Y has a finite subcover.

This means if

$$Y \subset \bigcup_{\alpha \in I} V_\alpha \qquad V_\alpha \in \tau$$

then there are finitely many $\alpha_1, \ldots, \alpha_n \in I$ s.t.

$$Y \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$$

Proposition 41.9 Let X be a compact top. space and $Y \subset X$ a closed subset. Then Y is compact.

Proof. Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of Y,

$$Y \subset \bigcup_{\alpha \in I} V_\alpha$$

Let $O = Y^c = X \setminus Y$. O is open, and $\{O, V_\alpha\}_{\alpha \in I}$ is an open cover of X.

$$X = O \cup \left(\bigcup_{\alpha \in I} V_\alpha \right)$$

Then there are finietly many $\alpha_1, \ldots, \alpha_n \in I$ s.t. $X = O \cup V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$.

$$Y^c \cup Y = Y^c \cup (V_{\alpha_1} \cup \ldots \cup V_{\alpha_n})$$

So, $Y \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$, and we have found a finite subcover. □

The converse fails.

Exercise 41.10 $X = \mathbb{R}$, $\tau = \{\varnothing, \mathbb{R}, O : O \subset \mathbb{R} : O^c \text{ is a finite set}\}$.

τ is Tychonoff but not Hausdorf. Let $O \subset \tau$ be any non-trivial open set ($O \neq \varnothing, O \neq X$). Show that O is compact in τ.

§42 Thursday, November 21, 2013

Proposition 42.1 Let (X, τ) be a compact Hausdorf topological space. Then if $Y \subset X$ is compact, Y is closed.

Proof. We want to show that Y^c is open. Take $x \in Y^c$. Since X is Hausdorf, there exists a neighborhood U_y of x and V_y of y s.t. $U_y \cap V_y = \varnothing$.

Now, $\{V_y\}_{y \in Y}$ is an open cover of Y ($y \in V_y$). Since Y is compact, there exists a finite subcover,

$$Y \subset V_{y_1} \cup \ldots \cup V_{y_n}$$

Set $U = U_{y_1} \cap \ldots \cap U_{y_n}$. Then U is a neighborhood of x, and $U \cap (V_{y_1} \cup \ldots \cup V_{y_n}) = \varnothing$. So, $U \cap Y = \varnothing$ and $U \subset Y^c$. So, for any $x \in Y^c$, there exists a neighborhood U of x s.t. $U \subset Y^c$. So Y^c is open, and Y is closed.

$\square$

Proposition 42.2 Let (X, τ) be a compact, topological, Hausdorf space. Then X is normal.

Proof. We want to show that if F_1, F_2 are two closed, non-empty, disjoint subsets of X, then there are open sets O_1 and O_2 s.t. $O_1 \supset F_1$, $O_2 \supset F_2$, $O_1 \cap O_2 = \varnothing$.

Suppose first that $F_1 = \{x\}$. We repeat the argument of previous proposition. Since X is Hausdorf, for any $y \in F_2$, we can find neighborhoods U_y of x and V_y of y s.t. $U_y \cap V_y = \varnothing$.

Now, $\{V_y\}_{y \in F_2}$ is an open cover of F_2, and F_2 is a closed subset of a compact topological space and so is compact. Then we have a finite subcover:

$$F_2 \subset V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$$

Set $O_1 = U_{y_1} \cap \ldots \cap U_{y_n}$ and $O_2 = V_{y_1} \cup \cdots \cup V_{y_n}$. Then O_1 is a neighborhood of x, O_2 is open, $O_2 \supset F_2$ and $O_1 \cap O_2 = \varnothing$.

Consider now general F_1. Then for any $x \in F_1$, we can find U_x, a neighborhood of x, and V_x, an open set that contains F_2, s.t.

$$U_x \cap V_x = \varnothing$$

$\{U_x\}_{x \in F_1}$ is an open cover of F_1. Then there exists a finite subcover,

$$F_1 \subset U_{x_1} \cup \ldots \cup U_{x_n}$$

Set $O_1 = U_{x_1} \cup \ldots \cup U_{x_n}$ and $O_2 = V_{x_1} \cap \ldots \cap V_{x_n}$.

O_1 is open, $O_1 \supset F_1$, O_2 is open, and $O_2 \supset F_2$. $O_1 \cap O_2 = \varnothing$, and we have found our sets.

$\square$

Proposition 42.3 Let (X, τ) be a compact topological space and $f : X \to Y$ a continuous function, where (Y, s) is a topological space. Then $f(X) = \{f(x) : x \in X\}$ is a compact subset of Y.

Proof. Let $\{V_i\}_{i \in I}$ be an open cover of $f(X)$ in Y, that is

$$f(X) \subset \bigcup_{i \in I} V_i$$

Then,

$$X = \bigcup_{i \in I} f^{-1}(V_1)$$

Since f is continuous, the inverse image of the open sets are open, so $\{f^{-1}(V_i)\}_{i\in I}$ is an open cover of X, and we can pick a finite subcover,

$$X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n})$$

$$f(X) = f(f^{-1}(V_{i_1})) \cup \cdots \cup f(f^{-1}(V_{i_n})) \subset V_{i_1} \cup \cdots \cup V_{i_n}$$

So, $V_{i_1}, \ldots, V_{i_n}$ is a finite subcover of $f(X)$. □

Proposition 42.4 Let (X, τ) be a compact topological space and $f : X \to \mathbb{R}$ a continuous function. Then f achieves its minimum and maximum values on X.

Proof. $f(X)$ is a compact subset of $\mathbb{R}$, so $f(X)$ is closed and bounded. Hence, $f(X)$ contains its infimum and supremum. □

If (X, d) is a metric space, then compactness is equivalent to sequential compactness. This fails for topological spaces.

If X is second countable, then compactness is equivalent to sequential compactness.

Remedy: the notion of "nets", or generalized sequences.

Aside This is a difference between fermions and bosons; fermions you always deal with compact spaces with "nice" properties, whereas with photons you do not.

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§42.1 Connected Topological Spaces

Definition 42.5 Let (X, τ) be a topological space. X is called connected if it cannot be written as a union of two disjoint open non-empty sets.

In other words, X is connected iff the relation

$$X = O_1 \cup O_2 \text{ where } O_1, O_2 \text{ are open and satisfy } O_1 \cap O_2 = \varnothing \implies O_1 = \varnothing \text{ or } O_2 = \varnothing$$

Definition 42.6 A subset $Y \subset X$ is called connected if the relation $Y \subset O_1 \cup O_2$ where O_1, O_2 are open and satisfy $Y \cap O_1 \cap O_2 = \varnothing$ implies $Y \cap O_1 = \varnothing$ or $Y \cap O_2 = \varnothing$.

Clarification: If (X, τ) is a top. space and $Y \subset X$, then the family of sets

$$s = \{Y \cap O : O \in \tau\}$$

is a topology for Y. (Y, s) is called a topological subpace. So a subset $Y \subset X$ is connected iff the top. space (Y, s) is connected.

Proposition 42.7 Let (X, τ) be a topological space. T.F.A.E.:

1. X is connected
2. $\varnothing$ and X are the only two subsets of X that are simultaneously open and closed

Proof. (1) $\implies$ (2) Let A be such that A is open and A^c is open. Then $X = A \cup A^c$, $A \cap A^c = \varnothing$. Since X is connected, either $A = \varnothing$ or $A^c = \varnothing$.

(2) $\implies$ (1): Suppose $X = O_1 \cup O_2$ where O_1, O_2 are open and $O_1 \cap O_2 = \varnothing$. Then $O_1 = O_2^c$. So O_1 is open and closed. So $O_1 = \varnothing$ or $O_1 = X$. So either $O_1 = \varnothing$ or $O_2 = \varnothing$, and X is connected. □

Proposition 42.8 Let (X, τ) and (Y, s) be top. spaces and assume that $f : X \to Y$ is a continuous function. Suppose that X is connected. Then

$$f(X) = \{f(x) : x \in X\}$$

is a connected subset of Y.

Proof. Suppose that $f(X) \subset V_1 \cup V_2$, where V_1, V_2 are open in Y and $V_1 \cap V_2 \cap f(X) = \varnothing$. Note that

$$X = f^{-1}(V_1) \cup f^{-1}(V_2)$$

and that $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$ (otherwise, if $x \in f^{-1}(V_1) \cap f^{-1}(V_2)$, then $f(x) \in V_1$ and $f(x) \in V_2$, and $f(X) \cap V_1 \cap V_2 \neq \varnothing$).

Since X is connected we must have that either $f^{-1}(V_1) = \varnothing$ or $f^{-1}(V_2) = \varnothing$.

$f^{-1}(V_1) = \varnothing \iff f(x) \cap V_1 = \varnothing$.

$f^{-1}(V_2) = \varnothing \iff f(X) \cap V_2 = \varnothing$.

So, we conclude that $f(X)$ is connected. □

Proposition 42.9 (Intermediate Value Theorem) Let (X, τ) be a connected topological space, and $f : X \to \mathbb{R}$ a continuous function. Let $a = \inf f(X)$, $b = \sup f(X)$, and let $a < r < b$. Then there exists a point $x \in X$ s.t. $f(x) = r$.

§43 Friday, November 22, 2013

Tutorial today: in Stratchona Anatomy and Dentistry Building SADB M-1.

Last class:

Proposition 43.1 (Intermediate Value Theorem) Let (X, τ) be a connected topological space, and $f : X \to \mathbb{R}$ a continuous function. Let $a = \inf f(X)$, $b = \sup f(X)$, and let $a < r < b$. Then there exists a point $x \in X$ s.t. $f(x) = r$.

Proof. Suppose that the statment is not true. Let $a < r < b$ be such that $f(x) \neq r$ for all $x \in X$.

Set $O_1 = (-\infty, r)$ and $O_2 = (r, \infty)$.

Then $f(X) \subset O_1 \cap O_2$, and $f(X) \cap O_1 \cap O_2 = \varnothing$. However, $f(X) \cap O_1$ is not an empty set, since $r > a = \inf f(x)$, r is not a lower bound for $f(X)$, and then $\exists\, x \in X$ s.t. $r > f(x)$.

Similarly, $f(X) \cap O_2 \neq \varnothing$, because $r < b = \sup f(X)$ and so there exists $x \in X$ s.t. $f(x) > r$.

It follows that the image is not connected. This contradicts the previous proposition. □

Proposition 43.2 Let (X, τ) be a topological space and $Y = \{0, 1\}$ with discrete metric. (So Y is a top. space with every subset open). T.F.A.E.:

1. X is connected
2. Any continuous function from X to Y is constant, that is, $f(x) = 0$ for all x, or $f(x) = 1$ for all x.

Proof. (1) $\implies$ (2). Let $f : X \to Y$ be a continuous function and $O_1 = f^{-1}(\{0\})$ and $O_2 = f^{-1}(\{1\})$. Then O_1, O_2 are open in X, $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \varnothing$. Since X is connected, $O_1 = \varnothing$ or $O_2 = \varnothing$. Hence, either $f(x) = 0$ for all x, or $f(x) = 1$ for all x.

(2) $\implies$ (1). $\neg(1) \implies \neg(2)$.

$\neg(1)$: X is not connected.

$\neg(2)$: There exists a continuous function $f : X \to Y$ that is not constant.

Since X is not connected, there exists open sets O_1, O_2 s.t. $X = O_1 \cup O_2$ but $O_1 \cap O_2 = \varnothing$ and O_1, O_2 are non-empty. Define $f(x) = \begin{cases} 1 & \text{if } x \in O_1 \\ 0 & \text{if } x \in O_2 \end{cases}$.

Then f is not the constant function because O_1 and O_2 are non-empty. f is continuous, since $f^{-1}(\{0\}) = O_1$ and $f^{-1}(\{1\}) = O_2$, so f is continuous and $\neg(2)$ holds. □

Proposition 43.3 Let (X_1, d_1) and (X_2, d_2) be metric spaces, and (X, d) their product. T.F.A.E.

1. X is connected
2. X_1 and X_2 are connected

Proof. $d((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$

(1) $\implies$ (2): Define projector maps

$$\pi_1((x_1, x_2)) = x_1, \qquad \pi_2((x_1, x_2)) = x_2$$

Each $\pi_i : X \to X_i$.

It is trivial to show that these are continuous.

Since X is connected, $X_1 = \pi_1(X)$ and $X_2 = \pi_2(X)$ are connected.

(2) $\implies$ (1): we will use previous proposition. Let $f : X \to Y = \{0, 1\}$ be a continuous function. We want to show that f is constant.

Given $x_1 \in X_1$, set $h_{x_1}(x_2) = f(x_1, x_2)$.

$h_{x_1} : X_2 \to \{0, 1\}$. h_{x_1} is a continuous function since f is. X_1 is connected, so h_{x_1} is a constant function. However, the constant depends on x_1.

Define now, for fixed $x_2 \in X_2$

$$g_{x_2} : X_1 \to \{0, 1\} \qquad g_{x_2}(x_1) = f(x_1, x_2)$$

Since X_1 is connected, g_{x_2} is a constant function (although the constant depends on x_2).

Let now $(x_1, x_2), (x_1', x_2') \in X$.

$$f(x_1, x_2) = h_{x_1}(x_2) = h_{x_1}(x_2') = f(x_1, x_2') = g_{x_2'}(x_1) = g_{x_2'}(x_1') = f(x_1', x_2')$$

So f is a constant.

□

Homework problem: $(X_n, d_n), n = 1, 2, \ldots$ and (X, d) their product,

$$X = \prod_{n=1}^{\infty} X_n$$

Then (X, d) is connected iff each (X_n, d_n) is connected.

Sketch: If X is connected, define a projection map $\pi_n : X \to X_n$, $\pi_n((x_n)_{k=1}^{\infty}) = x_n$. π_n is continuous, $\pi_n(X) = X_n$, so X_n is connected if X is connected.

For the converse, we cannot do as we did before because we have infinitely many spaces. Fix a point $a = (a_n)_{n=1}^{\infty}$ in $X = \prod_{n=1}^{\infty} X_n$.

For given $x = (x_n)_{n=1}^{\infty}$, define

$$x^{(k)} = (a_1, \ldots, a_k, x_{n+1}, x_{n+2}, \ldots)$$

Let now $f : X \to \{0, 1\}$ be a continuous function. Arguing as in the last proposition, show that

$$f(x) = f(x^{(k)})$$

for every k. As $k \to \infty$, $x^{(k)} \to a$. f is continuous, so $f(x^{(k)}) \to f(a)$. But I know $f(x) = f(x^{(k)})$ for every k. So $f(x) = \lim_{k\to\infty} f(x^{(k)}) = f(a)$. So f is a constant function.

Proposition 43.4 Let (X, τ) be a topological space and $\{A_i\}_{i\in I}$ a collection of connected subsets of X. Suppose that for any $i, j \in I$,

$$A_i \cap A_j \neq \varnothing$$

Then

$$A = \bigcup_{i\in I} A_i$$

is also connected.

Proof. Suppose A is not connected. Then there exist open sets O_1 and O_2 in X s.t.

$$A \subset O_1 \cup O_2 \quad (1)$$
$$A \cap O_1 \cap O_2 = \varnothing \quad (2)$$
$$A \cap O_1 \neq \varnothing \quad (3)$$
$$A \cap O_2 \neq \varnothing \quad (4)$$

$$A \cap O_1 = \left(\bigcup_{i \in I} A_i \right) \cap O_1 = \bigcup_{i \in I} (A_i \cap O_1) \neq \varnothing$$

Hence, there exist $i \in I$ s.t. $A_i \cap O_1 \neq \varnothing$. Similarly (4) implies that there exists $j \in I$ s.t. $A_j \cap O_2 \neq \varnothing$.

$i, j \in I$ are s.t.

$$A_i \cap O_1 \neq \varnothing, \qquad A_j \cap O_2 \neq \varnothing$$
$$A_i \subset O_1 \cup O_2, \qquad A_j \subset O_1 \cup O_2$$

by (1).

From (2), we also have,

$$A_i \cap O_1 \cap O_2 = \varnothing, \qquad A_j \cap O_1 \cap O_2 = \varnothing$$

By assumption, A_i, A_j are connected.

So, $A_i \cap O_2 = \varnothing$ and $A_j \cap O_1 = \varnothing$. That is, $A_i \subset O_1$ and $A_j \subset O_2$.

$$A_i \cap A_j = A_i \cap A_j \cap O_1 \cap O_2 \subset A \cap O_1 \cap O_2 = \varnothing$$

So, $A_i \cap A_j = \varnothing$, which is a contradiction. □

§44 Tuesday, November 26, 2013

Assignment due Tuesday, Dec. 3.

Last class: Basic result: $\{A_i\}_{i \in I}$ family of connected subsets of X s.t. $A_i \cap A_j \neq \varnothing$ for all i, j. Then $\bigcup_{i \in I} A_i$ is connected.

§44.1 Connected Components

Definition 44.1 Let (X, τ) be a top. space. A connected subset $C \subset X$ is called a connected component if for any connected subset $D \subset X$ such that $C \subset D$, we have $C = D$.

Proposition 44.2 A connected component is a closed set.

Proof. Let C be a connected component. Since C is connected, $\operatorname{cl} C$ is also connected. By definition of connected component then, since $C \subset \operatorname{cl} C$, we must have $C = \operatorname{cl} C$. □

Proposition 44.3 If C_1, C_2 are two connected components, then either $C_1 = C_2$ or $C_1 \cap C_2 = \varnothing$.

Proof. Suppose that $C_1 \cap C_2 \neq \varnothing$. Then by last class proposition, $C_1 \cup C_2$ is a connected set. Since $C_1 \subset C_1\ cup C_2$, we must have $C_1 = C_1 \cup C_2$. Since $C_2 \subset C_1 \cup C_2$ we must have $C_2 = C_1 \cup C_2$. So $C_1 = C_2$. □

Proposition 44.4 Let (X, τ) be a top. space. Then any point $x \in X$ is in some connected component of X.

Proof. Notice that singlets $\{x\}$ are always connected: Suppose that $\{x\} \subset O_1 \cup O_2$ and that $\{x\} \cap O_1 \cap O_2 = \varnothing$. Then $\{x\} \cap O_1 = \varnothing$ or $\{x\} \cap O_2 = \varnothing$.

Let now A be the collection of all connected sets that contain a given point x. A is non-empty since $\{x\} \in A$. Let now

$$C = \bigcup_{D \in A} D$$

Every element of the family A is connected, and intersection of any two elements of A is non-empty since x belongs to that intersection. Hence, by last class's proposition, C is a connected set. Obviously, $x \in C$.

Let now $\tilde{C}$ be a connected set s.t.

$$C \subset \tilde{C}$$

Then $x \in \tilde{C}$, $\tilde{C}$ is connected, so $\tilde{C} \in A$. Hence, $\tilde{C} \subset C$, and so $\tilde{C} = C$. Hence, C is a connected component. □

Given a topological space (X, τ), let $\{C_i\}_{i \in I}$ be the collection of all connected components. The sets C_i are closed, disjoint, connected, and

$$\bigcup_{i \in I} C_i = X$$

$\{C_i\}_{i \in I}$ is a decomposition of X into connected componenets.

Question: Are the connected components open? No, they don't have to be. Take $X = \mathbb{Q}$, with the usual metric topology. If $A \subset \mathbb{Q}$ and A has more than two points, then A is not connected. To see this, let $r_1, r_2 \in A$ with $r_1 < r_2$. Let i be an irrational number s.t. $r_1 < i < r_2$. Take $O_1 = (-\infty, i) = \{x \in \mathbb{Q} : x < i\}$, $O_2 = (i, \infty) = \{x \in \mathbb{Q} : x > i\}$. $A \subset O_1 \cup O_2 = \mathbb{Q}$. $A \cap O_1 \cap O_2 = \varnothing$. $A \cap O_1 = r_1$, and $A \cap O_2 = r_2$. So the only connected subsets of $\mathbb{Q}$ are singlets, $\{r\}$, and

$$Q = \bigcup_{r \in \mathbb{Q}} \{r\}$$

is the decomposition of $\mathbb{Q}$ into connected components. Each component is closed, but not open.

However, if (X, τ) has only finitely many connected components, then every connected component is open and closed. If $\{C_1, \dots, C_n\}$ are the connected components,

$$X = C_1 \cup \dots \cup C_n$$

and the $\{C_i\}$ are closed and disjoint. Then

$$C_1 = (C_2 \cup \dots \cup C_n)^c = C_2^c \cap \dots \cap C_n^c$$

is the finite intersection of open sets and so is open.

Remark. Empty set is not connected.

§44.1.1 Examples of connected sets

Example 44.5

1. Any point in topological space is a connected set.
2. It may happen that only connected subsets of a topological space are singlets (e.g. $\mathbb{Q}$). Such a space is called totally disconnected. Another example: X an arbitrary non-empty set, d the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then any subset of X is open and closed. Any set A that consists of more than one point is not connected. To see this, take $x \in A$ and define

$$O_1 = \{x\} \qquad O_2 = A \setminus \{x\}$$

3. Closed interval $[a,b]$ is a connected subset of $\mathbb{R}$ (compact, too).

   Suppose $[a,b]$ is not connected. Then there are open sets O_1, O_2 with $O_1 \cap O_2 \cap [a,b] = \varnothing$ and $[a,b] \subset O_1 \cup O_2$, but $[a,b] \cap O_1 \neq \varnothing$ and $[a,b] \cap O_2 \neq \varnothing$.

   Suppose that $a \in O_1$. Define $R = \{\varepsilon : 0 < \varepsilon < b-a \text{ and } [a, a+\varepsilon) \subset O_1\}$. First, R is non-empty: $a \in O_1$, O_1 open, so for some $\delta > 0$, $(a-\delta, a+\delta) \subset O_1$. We can always take $\delta < b-a$, and so $\delta \in R$.

   Let $\varepsilon_0 = \sup R$. Obviously, $\varepsilon_0 \leq b-a$.

   Claim 1. $[a, a+\varepsilon_0) \subset O_1$.

   Proof: Let $n > 0$ be an integer s.t. $n > \frac{1}{\varepsilon_0}$. Consider then $\varepsilon_0 - \frac{1}{n}$. Positive number, and smaller than ε_0. Then $\varepsilon_0 - \frac{1}{n}$ is not an upper bound of R and so we can find $\varepsilon' \in R$ s.t. $\varepsilon_0 - \frac{1}{n} < \varepsilon' < \varepsilon_0$. Then,

$$[a, a+\varepsilon_0 - 1/n) \subset [a, a+\varepsilon') \subset [a, a+\varepsilon_0)$$

   We also have that

$$[a, a+\varepsilon_0 - 1/n) \subset [a, a+\varepsilon') \subset O_1$$

$$\bigcup_{n > \frac{1}{\varepsilon_0}} [a, a+\varepsilon_0 - 1/n) = [a, a+\varepsilon_0) \subset O_1$$

   Claim 2. $a + \varepsilon_0 \notin O_1$. Suppose that $a + \varepsilon_0 \in O_1$. Two cases:

   Case 1. $a + \varepsilon_0 = b$. Then $[a,b] \subset O_1$. Then $[a,b] \cap O_2 = [a,b] \cap O_1 \cap O_2 = \varnothing$. Contradiction of the choice of O_1, O_2.

   Case 2. $a+\varepsilon_0 < b$. Then, we can find $\delta > 0$ s.t. $a+\varepsilon_0+\delta < b$, and $(a+\varepsilon_0-\delta, a+\varepsilon_0+\delta) \subset O_1$. But then $[a, a+\varepsilon_0+\delta) = [a, a+\varepsilon_0) \cup [a+\varepsilon_0, a+\varepsilon_0+\delta) \subset O_1$. So $\varepsilon_0+\delta \in R$, contradicting the choice of ε_0.

   Claim 3. $a + \varepsilon_0 \notin O_2$.

   Suppose that $a+\varepsilon_0 \in O_2$. Then there exists $\delta > 0$, $\varepsilon_0 > \delta > 0$ s.t. $(a+\varepsilon_0-\delta, a+\varepsilon_0+\delta) \subset O_2$. Take $r \in (a+\varepsilon_0-\delta, a+\varepsilon_0)$. Then $[a, r) \not\subset O_1$ since $(a+\varepsilon_0-\delta, r) \subset O_2$. On the other hand, $r < a+\varepsilon_0$ and $[a, a+\varepsilon_0) \subset O_1$. So, $[a, r) \subset [a, a+\varepsilon_0) \subset O_1$. Contradiction.

   Claims 2,3 state that $a + \varepsilon_0 \notin O_1$, $a + \varepsilon_0 \notin O_2$, but $\varepsilon_0 < b - a$, $a + \varepsilon_0 \in [a,b]$ and

$$[a,b] \subset O_1 \cup O_2$$

   Contradiction.

§45 Thursday, November 28, 2013

Last Class: $[a,b]$ is a connected subset of $\mathbb{R}$.

Proposition 45.1 Interval (a,b), where $-\infty \leq a < b \leq \infty$ is a connected subset of $\mathbb{R}$.

Proof. Consider first the case a and b finite, then

$$(a,b) = \bigcup_{n > \frac{2}{b-a}} [a + 1/n, b - 1/n]$$

If $A_n = [a + 1/n, b - 1/n]$, A_n's are connected and $\frac{b+a}{2} \in A_n \cap A_m$ (center of $[a,b]$). So A_n, $n > \frac{2}{b-a}$, is a family of connected sets such that any two elements of the family have non-empty intersection. Then, by our previous proposition,

$$\bigcup_{n > \frac{2}{b-a}} A_n \text{ is connected.}$$

If a is finite and $b = \infty$, one can write

$$(a, \infty) = \bigcup_{n=1}^{\infty} A_n, \text{ where } A_n = (a, a+n)$$

and conclude in the same way that (a, ∞) is connected. If b is finite and $a = -\infty$, write $(-\infty, b) = \bigcup_{n=1} A_n$ where $A_n = (b-n, b)$ and conclude the proof. Finally, if $a = -\infty$, $b = \infty$, write $(-\infty, \infty) = \bigcup_{n=2}^{\infty} A_n$ where $A_n = [-n, n]$ and conclude that $(-\infty, \infty)$ is connected. □

Exercise 45.2 Prove that the intervals $(a, b]$, $-\infty \le a < b < \infty$ and $[a, b)$ where $-\infty < a < b \le \infty$ are connected.

Exercise 45.3 Let $f : [0, 1] \to [0, 1]$ be a continuous function. Then there exists $x \in [0, 1]$ s.t. $f(x) = x$

Proof. We know $[0, 1]$ is connected and f is continuous. Set $h(x) = f(x) - x$. h is continuous on $[0, 1]$, $h(0) = f(0) \ge 0$ and $h(1) = f(1) - 1 \le 0$. If $h(0) = 0$ then we know $f(0) = 0$; If $h(1) = 0$, then $f(1) = 1$. Otherwise, if $f(0) > 0$ and $h(1) < 0$, we have that

$$a = \inf\{h(x) : x \in [0, 1]\} < 0$$

$$b = \sup\{h(x) : x \in [0, 1]\} > 0$$

By intermediate value theorem, h takes all the values in the interval (a, b) and so $\exists x \in [0, 1]$ s.t. $h(x) = 0$, or $f(x) = x$. □

Proposition 45.4 Let $O \subset \mathbb{R}$ be a non-empty open connected set. Then $O = (a, b)$ for some $-\infty \le a < b \le \infty$.

Proof. Let $a = \inf O$, $b = \sup O$.

Claim: $a, b \notin O$. If $a = -\infty$, then $a \notin O$. If a is finite and $a \in O$, then since O is open, there exists $\delta > 0$ s.t. $(a - \delta, a + \delta) \subset O$, but then, there is an $x \in O$ s.t. $x < a$, and a is NOT a lower bound. So $a \notin O$, and in the same way one shows that $b \notin O$. Obviously, $O \subset (a, b)$, (if $x \in O$, then $a < x < b$). We need to prove that $(a, b) \subset O$. Suppose it is not so, namely that there exists $r \in (a, b)$ such that $r \notin O$. Let $O_1 = (-\infty, r)$, $O_2 = (r, \infty)$. Then, $O \subset O_1 \cup O_2$ and $O \cap O_1 \cap O_2 = \varnothing$. However, $O \cap O_1 \neq \varnothing$, since $r > a$, and r is not lower bound for O. In the exact same way, $O \cap O_2 \neq \varnothing$. But then O is not connected, contradiction. □

Exercise 45.5 Using similar argument, prove that if A is a connected subset of $\mathbb{R}$, then

$$A = (a, b) \text{ or } A = [a, b), \text{ or } A = (a, b] \text{ or } A = [a, b]$$

HINT: Set $a = \inf A$, $b = \sup A$. For any $x \in A$, $a \le x \le b$. $A \subset [a, b]$. Then, show that $(a, b) \subset A$ by repeating the argument of the previous proposition.

What about $\mathbb{R}^n$? If $A_1, \cdots, A_n$ are connected subsets of $\mathbb{R}$, then $A_1 \times \cdots \times A_n$ is a connected subset of $\mathbb{R}^n$.

Example 45.6 $\mathbb{Q} \times \mathbb{Q}$ is not a connected subset of $\mathbb{R}^2$. $O_1 = \{(x, y) \in \mathbb{Q} \times \mathbb{Q}, y < \pi\}$, $O_2 = \{(x, y) \in \mathbb{Q} \times \mathbb{Q}, y > \pi\}$ these two open sets separate $\mathbb{Q} \times \mathbb{Q}$.

Example 45.7 Let $A = \{(x, y) \in \mathbb{R}^2 :$ either x or y is rational$\}$ this is a connected subset of $\mathbb{R}^2$. (it is path connected)

§45.1 Path Connectedness (only for metric spaces)

Definition 45.8 Let (X, d) be a metric space. A continuous map $f : [0, 1] \to X$ is called a path in X. $f(0) = x_0$ is the initial point of the path, $f(1) = x_1$ is the final point, and the path is called a loop iff the beginning and end point are the same. A path is called a loop if $x_0 = x_1$. Two points x and y are called path connected if there exists a path $f : [0, 1] \to \mathbb{R}$ such that $f(0) = x$, and $f(1) = y$.

Relation: $x \sim y \iff x$ and y are path connected. $\sim$ is a relation of equivalence. $x \sim x$. $f(t)$, $t \in [0,1]$, $x \sim y$, and f is a path connecting x and y, then $g(t) = f(1-t)$ is a path connecting y and x, so $y \sim x$. Suppose that $x \sim y$ and $y \sim z$. Then let f be a path connecting x and y and g a path connecting y and z. Then define

$$h(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

If $t_n \to \frac{1}{2}$, then $h(t_n)$ is either $f(2t_n)$ or $g(2t_n - 1)$ "Write Clear Proof." So h is a path that connects x and z.

Definition 45.9 A metric space (X,d) is called path connected if any two points in X can be connected by a path. (similar in general topology but more complicated with continuity) A subset $Y \subset X$ is called path connected if the metric subspace (Y,d) is path connected. In other words, Y is path connected if any two points $x, y \in Y$ can be connected by a path that lies entirely in Y.

Example 45.10 Let $(X, \|\cdot\|)$ be a normed space and $B(x_0, r)$ a ball of radius $r > 0$ centered at x_0. Then $B(x_0, r)$ is path connected.

Proof. Let $x, y \in B(x_0, r)$ and $f(t) = (1-t)x + ty$, $t \in [0,1]$ which connects points x and y. f is a path connecting x and y, and we need to check that this path lies entirely in $B(x_0, r)$. $\|f(t) - x_0\| = \|(1-t)x + ty - x_0\| = \|(1-t)x + ty - [(1-t)x_0 + tx_0]\| = \|(1-t)(x - x_0) + t(y - y_0)\| \leq (1-t)\|x - x_0\| + t\|y - y_0\| < (1-t)r + tr = r$. □

Proposition 45.11 Suppose that the metric space (X,d) is path connected. Then (X,d) is connected.

Proof. Let $x \in X$ be fixed. For any $y \in X$ there is a path $f_y : [0,1] \to X$ connecting x and y. Set $U_y = f_y([0,1]) = \{f_y(t) : t \in [0,1]\}$. Then U_y is connected since f_y is continuous and $[0,1]$ is connected. $y \in U_Y$ so $\bigcup\limits_{y \in X} U_y = X$. For any $y_1, y_2 \in X$, $x \in U_{y_1} \cap U_{y_2}$, since $x \in U_y$ for all y. So, X is connected. □

§46 Friday, November 29, 2013

Last class: Prop: Let (X,d) be path connected metric space. Then (X,d) is connected.

If $X = \mathbb{R}$, the converse is true. If a subset $A \subset \mathbb{R}$ is connected, then A is an interval ($[a,b]$,(a,b),$[a,b)$, $(a,b]$) and is path connected. If $x, y \in A$ then $f(t) = (1-t)x + ty$ connects x and y. In $\mathbb{R}^2$, a subset can be connected but not path connected. Examples on the assignment:

$$A \subset \mathbb{R}^2, A = \{(x, \sin {}^1\!/\!_x, 0 \leq x \leq {}^2\!/\!_\pi)\} \cup \{(0,y) : -1 \leq y \leq 1\}$$

Proposition 46.1 Let $(X, \|\cdot\|)$ be a metric space and let $O \subset X$ be open and connected. Then O is path connected.

Proof. Let $x \in O$ be given and let $O_1 = \{y \in O : x \sim y\}$ ($x \sim y \iff x$ and y path connected.) O_1 is non-empty ($x \in O_1$) and open: let $y \in O_1$ and $r > 0$ s.t. $B(y,r) \subset O$. Now if $z \in B(y,r)$, $y \sim z$ (our first example: $f(t) = (1-t)y + tz$). Since $x \sim y$ and $y \sim z$, we have $x \sim z$. Thus, $B(y,r) \subset O_1$, and hence O_1 is open.

Suppose that $O_1 \neq O$. This means there is a point $y \in O$ that is not connected to x. Let then

$$O_2 = \{y \in O_1 : y \not\sim x\}$$

O_2 is non-empty; it is disjoint from O_1 and we claim that O_2 is also open. Let $y \in O_2$. Let $r > 0$ be such that $B(y,r) \subset O$. Then for any $z \in B(y,r)$, we have $y \sim z$. Now, if $x \sim z$, then $x \sim z \sim y$, and we would have $x \sim y$, so we must have $x \not\sim z$. So for any $z \in B(y,r)$, $z \in O_2$. So O_2 is open and $O = O_1 \cup O_2$, $O_1 \cap O_2 = \varnothing$, with both O_1 and O_2 non-empty. This contradicts that O is connected, and so we must have that O_2 is empty. Hence, $O = O_1$ which is path-connected. □

§46.1 Fundamental group (not on exam)

Let (X, d) be a path-connected metric space. Fix $x \in X$ and consider the collection of all loops $f : [0,1] \to X$, $f(0) = f(1) = x$. We say that two loops are homeotopically equivalent if there exists a continuous map

$$\begin{aligned} H : [0,1] \times [0,1] \to X \text{ s.t.} \\ H(t,0) = f(t) : t \in [0,1] \\ H(t,1) = g(t) : t \in [0,1] \\ H(0,s) = H(1,s) = x_0 : \text{ for } s \in [0,1] \end{aligned}$$

for any fixed $s \in [0,1]$.

The map $t \to H(t,s)$ is a loop; $s = 0$, the loop is $f(t)$; $s = 1$, the loop is $g(t)$. H is a continuous deformation of the loop f to the loop g in X.

Introduce the relation $f \sim g \iff f, g$ are homeotopically equivalent. This is a relation of equivalence on the set of all loops with the same basepoint x_0. Consider classes of equivalences,

§47 December 13, 2013 - Review Session

§47.1 Problem 2

(X, d) a metric space, $F \subset X$ closed. Show that there exists a sequence of continuious functions f_n s.t. $0 \le f_n \le 1$, and $\lim_{n\to\infty} f_n = \chi_F(x)$ for $x \in F$.

Solution: Let

$$F_n = \{x \in X : d(x,F) \ge \frac{1}{n}\}$$

Then $d(x,F)$ is continuous, F_n is closed, and $F_n \cap F = \varnothing$, since $d(x,F) = 0 \iff x \in F$ (since F is closed).

Then, by previous problem,

$$f_n(x) = \frac{d(x,F_n)}{d(x,F) + d(x,F_n)}$$

satisfies $0 \le f_n(x) \le 1$, $f_n(x) = 0 \iff x \in F_n$, $f_n(x) = 1 \iff x \in F$.

We want to show that

$$\lim f_n = \chi_F$$

That is,

$$\lim f_n(x) = 0 \text{ if } x \in F^C$$

If $x \in F^c$, there is $N > 0$ (depending on x) s.t. $d(x,F) \ge \frac{1}{N}$. BUt then $x \in F_n$ for $n \ge N$. So $f_n(x) = 0$ for $n \ge N$, so $\lim_{n\to\infty} f_n - 0$.

§47.2 Problem 3

Consider $C([a,b])$ and let $S \subset C([a,b])$ consirts of all functions f s.t.

1. f is continuous and differentabile on $[a,b]$ and $|f'(x)| \le 1$ for $x \in [a,b]$.
2. There exists a least one $x \in [a,b]$ s.t. $f(x) = 0$.

Show that $\operatorname{cl} S$ is a compact subset of $C([a,b])$.

Solution: S is bounded: Want to show that $\|f\| \leq M$ for all $f \in S$.

For fixed f, choose x_0 so $f(x_0) = 0$.

For all $x \in [a, b]$,

$$|f(x)| = |f(x) - f(x_0)| = \left| \int_{x_0}^{x} f'(\tau)\,\mathrm{d}\tau \right| \leq \int_{x_0}^{x} |f'(\tau)|\,\mathrm{d}\tau \leq \int_{x_0}^{x} 1\,\mathrm{d}\tau \leq b - a$$

So for every $f \in S$, $\|f\| \leq b - a$.

Addtionally, S is equicontiunous: fix $\varepsilon > 0$. Then for all $x, y \in [a, b]$ s.t. $|x - y| < \delta = \varepsilon$, by the mean value theorem $\exists\, c \in (a, b)$ s.t.

$$|f(x) - f(y)| = |f'(c)||x - y| \leq |x - y| < \varepsilon$$

§47.3 Problem 4

$f \in C([a, b])$ and there exists a sequence of polynomials p_k of bounded degree, $\deg p_k \leq n$, s.t.

$$p_k \to f$$

in $C([a, b])$, as $k \to \infty$.

Show that f is also a polynomial of degree $\leq n$.

Solution: Polyomials of bounded degree form a finite dimensional vector subspace of $C([a, b])$.

We know that if X is a normed space and $Y \subset X$ is a finite dimensional subspace, then Y is closed.

$X = C([a, b])$, $Y = \operatorname{span}\{1, x, \ldots, x^n\}$. $\dim Y = n + 1$.

$1, x, \ldots, x^n$ are linearly dependent if there exists $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$, not all zero, such that $\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n = 0$ (the zero function). This is not possible; such a polynomial has n zeros (possibly complex).

So Y is closed, and thus $f \in Y$. Thus, Y is a polynomial of dimension at most $n + 1$.

§47.4 Problem 5

$C([0, 1])$. $S = \{x^n : n \geq 0\}$. Is $\mathrm{cl}(S)$ a compact subset of $C([0, 1])$?

Solution: No, it is not compact. Consider the sequence $f_n(x) = x^n \in S$. This sequence does not have a convergent subsequence in the supremum norm.

If f_{n_k} would be a convergent subsequence, say $f_{n_k} \to g$ in $C([0, 1])$, then $f_{n_k}(x) \to g(x)$ as $k \to \infty$ for all $x \in [0, 1]$.

But $\lim_{k\to\infty} x^{n_k} = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}$.

So $g(x)$ is not continuous; contradiction.

Since $f_n \in \mathrm{cl}\, S$, $\mathrm{cl}\, S$ cannot be compact.

§47.5 Problem 6

$C([0, 1])$, $S = \{\sin(nx) : n \geq 0\}$. Suppose there exists subsequence $\sin(n_k x)$ that converges uniformly on $[0, 1]$ to a continuous function $g(x)$.

That would mean

$$(\forall\, \varepsilon > 0)(\exists\, N > 0)(\forall\, k \geq N) \left(\max_{x \in [a,b]} |\sin(n_k x) - g(x)| < \varepsilon \right)$$

$g(0) = 0$; $|g(0)| < \varepsilon$ for all $\varepsilon > 0$. Set $x_k = \frac{\pi}{2n_k}$.

Then
$$\left|1 - g\left(\frac{\pi}{2n_k}\right)\right| < \varepsilon$$
Send $k \to \infty$,
$$|1 - g(0)| < \varepsilon$$
Contradiction.

§47.6 Problem 7

$C([0,1])$, $S = \{\sin(n+x) : n \geq 0\}$. Is $\operatorname{cl} S$ compact?

By MVT, $\exists\, c \in (x+n, y+n)$ s.t.
$$|\sin(x+n) - \sin(y+n)| = |\cos c||x+n-(y+n)| \leq |x-y|$$
This implies equicontinuity. Since S is bounded, $\operatorname{cl} S$ is compact.

§47.7 Problem 8

$C([0,1])$, $S = \{\arctan(nx) : n \geq 0\}$. Is $\operatorname{cl} S$ compact?
$$\lim_{n\to\infty} \arctan(nx) = \begin{cases} 0, & x = 0 \\ \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$$
So the pointwise limit is not continuous.

§47.8 Problem 9

(X, d) a metric space, $f : X \to X$ continuous and onto. Let $S \subset X$ be dense. Show that $f(S) = \{f(x) : x \in S\}$ is also dense in X.

Since S is dense, $O \cap S$ is nonempty for any open set O.

Let $O \subset X$ be open. Then $f^{-1}(O)$ is open (and non-empty because f is surjective), so $S \cap f^{-1}(O)$ is non-empty. Then $f(S \cap f^{-1}(O)) = f(S) \cap O$ is non-empty, so $f(S)$ is dense.

§47.9 Problem 10

G_δ set: countable intersection of open set.

Let X be a metric space and $f : X \to \mathbb{R}$ a bounded function. Let $S = \{x \in X : f \text{ is continuous at } x\}$. Show that S is G_δ.

Define $v_k(x) = \sup_{x \in B(x, 1/k)} f(x) - \inf_{x \in B(x,1/k)} f(x)$. Clearly, $v_k(x) \geq 0$. Notice that $v_{k+1}(x) \leq v_k(x)$. So $\lim_{k\to\infty} v_k(x)$ exists; call this limit $v(x)$.

Show f is continuous at $x \iff v(x) = 0$.

For $n \in \mathbb{N}$, set $O_n = \{x \in X : v(x) < \frac{1}{n}\}$. Show that O_n is open.

Then show that $S = \bigcap_{n=1}^{\infty} O_n$.

§48 Glossary