Is QSEP-CIRCUIT QMA-hard?

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May 2013

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Maybe!

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- QMA is a complexity class that is the quantum analogue of NP
- QSEP-CIRCUIT is the problem of determining whether not a state is separable (not entangled), when given the state as a quantum circuit

Outline

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What is a complexity class?

Example

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What is a complexity class?

- A set of problems that can be solved with an amount of resources that scale with the input in a certain way

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Definitions and notation

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  - The set of valid characters for a string in a language L is called the *alphabet* associated to the L
- A *polynomial time verifier* is an algorithm that takes in an input x and a string y, and outputs "yes" or "no" in time polynomial in $|x|$

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- Furthermore, we'll require that every string where the first character is n, the second character is $=$, and the third character is any integer $z \in \mathbb{Z}$ greater than or equal to 5 is in the language L, and no other strings are in L.

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- Furthermore, we'll require that every string where the first character is n, the second character is $=$, and the third character is any integer $z \in \mathbb{Z}$ greater than or equal to 5 is in the language L, and no other strings are in L.
  - So "$n = 6$" $\in L$, but "abc" $\notin L$, "$n = 2$" $\notin L$, and "$= n6$" $\notin L$

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Example

Factoring: Given a huge number N with $1 \leq M \leq N$, (say, RSA-210: $N = 245246644900278211976517663573088018467026787678332759743414451715061600830038587216952208399332071549103626827191679864079776723243005600592035631246561218465817904100131859299619933817012149335034875870551067$) does N have a factor d with $1 < d < M$? Very difficult to do in general. But given a prospective factor, just divide to check.

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MA

- MA is the probabilistic equivalent of NP; it is the set of decision problems whose solutions can be checked in polynomial time by a probabilistic Turing machine

Definition

- A *probabilistic Turing machine* is simply a Turing machine that can use randomness and is allowed to have some (small) probability of error

MA

- MA is the probabilistic equivalent of NP; it is the set of decision problems whose solutions can be checked in polynomial time by a probabilistic Turing machine
- That is, MA is the set of decision problems L where there exists a probabilistic polynomial time verifier V and polynomial p such that: for instances of the problem x that have "yes" answers ($x \in L$), then there exists a proof y with $|y| \leq p(|x|)$ where $V(x, y) =$ yes with probability at least $2/3$. For instances x with a "no" answer ($x \notin L$), for any proof y with $|y| \leq p(|x|)$, $V(x, y) =$ no with probability at least $2/3$.

Definition

- A *probabilistic polynomial time verifier* is a polynomial time verifier that is allowed to use randomness

NP

- Or, NP is the set of decision problems whose "yes" instances have solutions that can be checked in polynomial time by a deterministic Turing machine
- Alternatively, NP is the set of decision problems ("languages", L) where there exists a polynomial time verifier V and polynomial p such that: for instances of the problem x that have "yes" answers ($x \in L$), then there exists a proof y with $|y| \leq p(|x|)$ where $V(x,y) = \text{yes}$. For instances x with a "no" answer ($x \notin L$), for any prospective proof y with $|y| \leq p(|x|)$, $V(x,y) = \text{no}$.

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Reductions

- A problem is called "hard" for a complexity class if it can be reduced to any other problem in the class
- A problem is called complete if it is in the class and hard for the class
- If problem A can be reduced to problem B, that is, if we can solve A by assuming we can solve B, then we say $A \leq B$

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- To find the first hard problem, usually the definition of a class is recast as an automatically-hard problem

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- We will consider two promise problems: one called called "$\text{WMEM}_{\delta'}(K)$", the weak membership problem with two-sided error, and one called "$\text{WMEM}^1_{\delta}(K)$", the weak membership problem with one-sided error.

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- The idea is that if we have certain type of set of points called a convex set, and we know if a point we ask about, y, is either deep inside the set, or far away from the set, we can make an algorithm that does even better: our algorithm will tell us if a point is inside the set (even if it's close to the boundary), or far from the set.

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- In this way, we've reduced the weak membership problem with one-sided error to the weak membership problem with two-sided error – in effect, we've shown that getting rid of half the error doesn't make the problem any easier!

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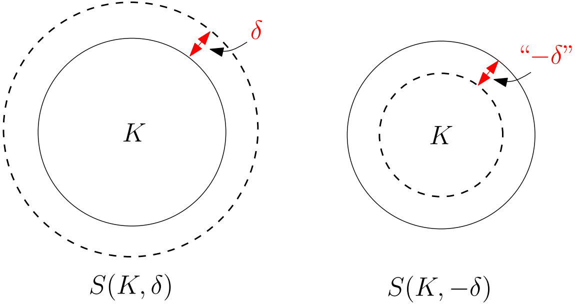
- The interior of K with radius ε is $S(K, -\varepsilon) = \{x \in \mathbb{R}^n : B(x, \varepsilon) \subseteq K\}$

What do these sets look like?



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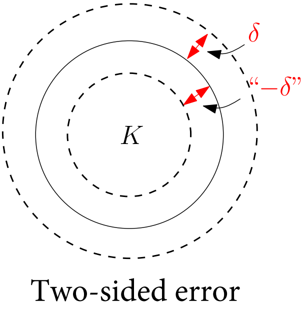

The Algorithm

- Goal: Solve WMEM with 1-sided error $\delta > 0$ by assuming we can solve WMEM with 2-sided error $\delta' = \frac{r}{4R}\delta$.

Reminder

- $\delta' = \frac{r\delta}{4R}$
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  - In that case, $2R \leq \|y - p\| \leq \|y\| + \|p\| \leq \|y\| + R$, so $R \leq \|y\|$

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- Otherwise, solve $\mathrm{WMEM}(K)_{\delta'}$ for the point $y' = \left(1 - \frac{\delta}{4R}\right) y + \left(\frac{\delta}{4R}\right) p$, and output the answer.

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Case: $WMEM_{\delta'}$ outputs "YES" for y'

- In this case, we know $y' \in S(K, \delta')$. (Either could have output "YES" because $y' \in S(K, -\delta')$, or because y' is not a valid input and is too close to the border).

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- $\|y - x\| = \|(y' - x) + (y - y')\| \leq \|y' - x\| + \|y - y'\| < \delta' + \delta/2 < \delta/4 + \delta/2 < \delta$
- So $y \in S(K, \delta)$, and we were correct to output "YES"

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- We know for any $d \in \mathbb{R}^n$ with $\|d\| < r$ that $p + d \in K$ (because $B(p, r) \subseteq K$)

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- So $\alpha := \left(1 - \frac{\delta}{4R}\right) y + \left(\frac{\delta}{4R}\right) (p + d) \in K$, because this point lies on the line between y and $p + d$, and K is convex

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- $B(p, r) \subseteq K \subseteq B(0, R) \subseteq \mathbb{R}^n$
- $y' = \left(1 - \frac{\delta}{4R}\right) y + \left(\frac{\delta}{4R}\right) p$

Case: $WMEM_{\delta'}$ outputs "NO" for y'

- Then we know that $y \notin (K, -\delta)$. We want to show then that $y \notin K$, so assume $y \in K$ to try to reach a contradiction
- We know for any $d \in \mathbb{R}^n$ with $\|d\| < r$ that $p + d \in K$ (because $B(p, r) \subseteq K$)
- So $\alpha := \left(1 - \frac{\delta}{4R}\right) y + \left(\frac{\delta}{4R}\right)(p + d) \in K$, because this point lies on the line between y and $p + d$, and K is convex
- $\alpha = y' + \frac{\delta}{4R} d \in K$, so $B(y', \frac{r\delta}{4R}) \subseteq K$, i.e. $y \in S(K, -\delta')$

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What have we done and why?

- We've shown that weak membership with 1-sided error is just as hard as weak membership with 2-sided error by a reduction, by creating an algorithm that assumes we can solve weak membership with 2-sided error in order to solve weak membership with 1-sided error

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- We've shown that weak membership with 1-sided error is just as hard as weak membership with 2-sided error by a reduction, by creating an algorithm that assumes we can solve weak membership with 2-sided error in order to solve weak membership with 1-sided error
- The set of separable states is convex, so this is actually relevant to QSEP-CIRCUIT

Outline

Interactive Proof Systems

- An interactive proof system models computation as the exchange of messages between a “prover” and “verifier”.

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- and completeness: if the statement is false, the prover will only have a small probability of convincing the verifier

NP and MA (again)

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- NP is the set of problems that can be solved by a computationally unbounded prover sending a single polynomial-length message to a polynomial-time deterministic verifier such that if the prover is given a "yes" instance, he can always send a message that is accepted by the verifier as a correct proof, and if he is given a "no" instance, any message the prover sends will be rejected.

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- MA is the set of problems that can be solved by a computationally unbounded prover ("Merlin") sending a single polynomial-length message to a polynomial-time probabilistic verifier ("Arthur") such that if the prover is given a "yes" instance, he can always send a message that is accepted by the verifier as a correct proof with probability $\geq 2/3$, and if he is given a "no" instance, any message the prover sends will be rejected with probability $\geq 2/3$.

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Postulate 1: Quantum States

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- We ignore "global phases" - multiplying a state by $e^{i\theta}$ doesn't make a new state
- We don't care about the vector's length, so we make the convention that states are unit vectors (length 1)

More notation!

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- A quantum proof is a quantum state $|y\rangle$
- The size of a quantum proof $|y\rangle$, denoted $||y\rangle|$ is the number of qubits: log base 2 of the dimension of the vector
- A *quantum polynomial time verifier* is an algorithm that in an input state $|x\rangle$ and a quantum proof, a state $|y\rangle$, is allowed to use randomness, and outputs "yes" or "no" in time polynomial in $||x\rangle|$.

Outline

QMA

- QMA is just the quantum analogue of MA: it is the set of promise problems whose solutions can be checked in polynomial time by a quantum, probabilistic Turing machine.

Reminder

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- That is, the set of promise problems L where there exists a quantum polynomial time verifier V and polynomial p such that: for instances of the problem $|x\rangle$ that have "yes" answers ($x \in L$), then there exists a proof $|y\rangle$ with $||y\rangle| \leq p(||x\rangle|)$ where $V(|x\rangle, |y\rangle) =$ yes with probability at least $2/3$. For instances x with a "no" answer ($x \notin L$), for any proof $|y\rangle$ with $||y\rangle| \leq p(|x|)$, $V(|x\rangle, |y\rangle) =$ no with probability at least $2/3$.

Reminder

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MA

- MA is the probabilistic equivalent of NP; it is the set of decision problems whose solutions can be checked in polynomial time by a probabilistic Turing machine
- That is, MA is the set of decision problems L where there exists a probabilistic polynomial time verifier V and polynomial p such that: for instances of the problem x that have "yes" answers ($x \in L$), then there exists a proof y with $|y| \leq p(|x|)$ where $V(x, y) = \text{yes}$ with probability at least $2/3$. For instances x with a "no" answer ($x \notin L$), for any proof y with $|y| \leq p(|x|)$, $V(x, y) = \text{no}$ with probability at least $2/3$.

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Easily generalized: QIP(k)

- QIP(k) is the set of promise problems that can be solved by a computationally unbounded quantum prover exchanging k polynomial-length messages with a polynomial-time quantum verifier such that if the prover is given a "yes" instance, he can always send a message that is accepted by the verifier as a correct proof with probability $\geq 2/3$, and if he is given a "no" instance, any message the prover sends will be rejected with probability $\geq 2/3$.

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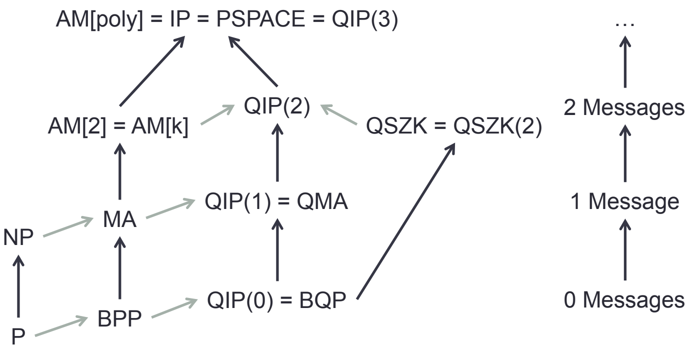
Theorem

PSPACE = QIP = QIP(3)

QSZK

- QSZK stands for quantum statistical zero-knowledge. It is just like QIP, except that for a "yes" instance, the verifier has to be able to generate the proof on his own: that is, the proof must be able to be generated in polynomial time

Hierarchy of Complexity Classes



What do we know about quantum complexity classes?

- We know several QMA-complete problems (unlike the thousands of known NP-complete problems)
- We know very little about QIP(2); the only QIP(2)-complete problem we know of is the trivial rewriting of the definition
- We know a lot about QIP(3) $=$ QIP because it equals PSPACE

Outline

Postulate 2: Time evolution

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- Unitary matrices are those that preserve inner product; in $\mathbb{R}^3$ for example, they are rotations and reflections.

Postulate 3: Measurement

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- After measurement, the system is in the state $\dfrac{M_j|\varphi\rangle}{\sqrt{\langle\varphi|M_j^\dagger M_j|\varphi\rangle}}$ (which is just $M_j|\varphi\rangle$ normalized to be a unit vector).

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- Subscripts or superscripts are used to denote which space a state lives in (e.g., $|\varphi\rangle_{AB}, |\varphi\rangle^{AB}$)

Density operators

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- A density operator ρ is separable iff it can be written as $\rho = \sum_i p_i \sigma_i \otimes \omega_j$ for $p_i \geq 0$, $\sum p_i = 1$, and density operators σ_i, ω_j. Otherwise it is entangled.

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- Fact: $\| * \|_{1-LOCC} \leq \| * \|_1$

Outline

Quantum Gates

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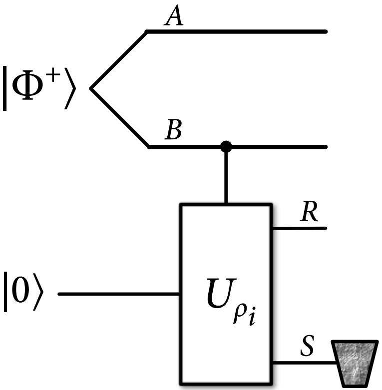
  - The number of qubits it acts on is $\log_2 n$, so a quantum gate that acts on a single qubit is simply a 2×2 unitary matrix.
- A quantum circuit is a sequence of quantum gates connected by "wires"

Quantum Circuits

Example



How to throw away one system?

- The partial trace over the B system of a density operator $\rho_{AB} = |a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|$, is $\rho_A = \mathrm{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \mathrm{tr}\left(|b_1\rangle\langle b_2|\right)$

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- Requiring linearity completes the definition

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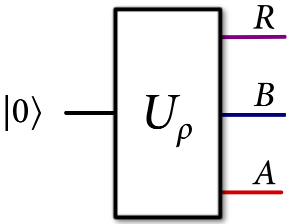
Example

$|0\rangle \in R \otimes A \otimes B$, want to generate $\rho_{AB} = \mathrm{tr}_R(U_\rho |0\rangle)$



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- Formally, QSEP-CIRCUIT(δ_c,δ_s) is: Given a quantum circuit to generate an n-qubit state ρ_{AB} and a labelling of qubits in the reference system R and the output qubits for A and B, decide whether

$$\min_{\sigma_{AB}\in S} \|\rho_{AB} - \sigma_{AB}\|_1 \leq \delta_c$$

or

$$\min_{\sigma_{AB}\in S} \|\rho_{AB} - \sigma_{AB}\|_{1-LOCC} \geq \delta_s$$

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- Specifying the state as a circuit has some advantages:
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  - More reasonable scaling: polynomial in the number of qubits the gates act on is exponential in the number of entries of a density matrix
    - So the quantum separability problem considers states much smaller than those considered by QSEP-CIRCUIT, which is why QSEP-CIRCUIT is a harder problem

NP-hardness of QSEP-CIRCUIT

- Can simply reduce to the quantum separability problem: if we could solve QSEP-CIRCUIT (figure out if big states are separable) then we could solve the quantum separability problem (figure out if small states are separable).

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- The quantum separability problem is already NP-hard, so QSEP-CIRCUIT is NP-hard as well

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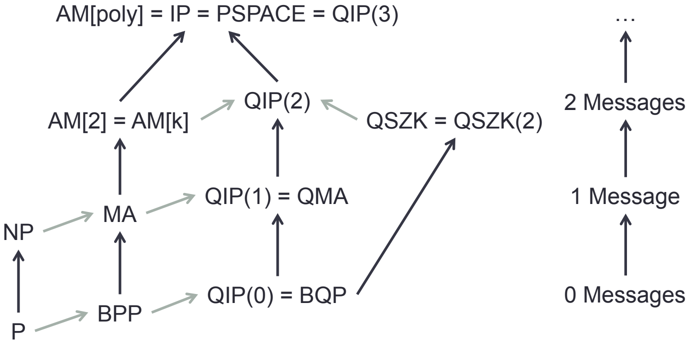
What do we know about QSEP-CIRCUIT?

- It is in QIP(2)
- It is NP-hard
- It is QSZK-hard
- A slightly modified problem about channel separability instead of state separability is QIP-complete

Hierarchy of Complexity Classes



So why do we care if QSEP-CIRCUIT is QMA-hard?

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- It would strongly suggest QSEP-CIRCUIT is QIP(2)-complete, because we think QSZK and QMA are separate classes and QIP(2) includes both
- It seems easier to prove than QIP(2)-hardness: QMA is the analogue of NP, so maybe we can find an analogous proof

Proof that the quantum separability problem is NP-hard:

CLIQUE $\leq_m$ WMQS $\leq_m$ RSDF $\leq_m$ $\mathrm{WOPT}_{\varepsilon}(\mathcal{S}_{M,N}) \leq_T$ $\mathrm{WMEM}^1_{\delta}(\mathcal{S}_{M,N}) \leq_m \mathrm{WMEM}_{\beta}(\mathcal{S}_{M,N})$

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- QSEP-CIRCUIT is the problem of determining whether or not a state is separable when given a circuit specifying the state
- Quantum information theory is fun!