Landauer's Principle

Eric Hanson

Departments of Math and Physics
McGill University

CUMC, July 2014

Outline

Outline

Landauer's principle

- The energy cost of erasure of one bit of information by the action of a thermal reservoir in equilibrium at temperature T is never less than $k_B T \log 2$.
  - What is erasure? An operation that takes any initial state ρ_{i} to a given final state ρ_{f}.

Landauer's principle

- The energy cost of erasure of one bit of information by the action of a thermal reservoir in equilibrium at temperature T is never less than $k_B T \log 2$.
  - What is erasure? An operation that takes any initial state ρ_{i} to a given final state ρ_{f}.

The setup

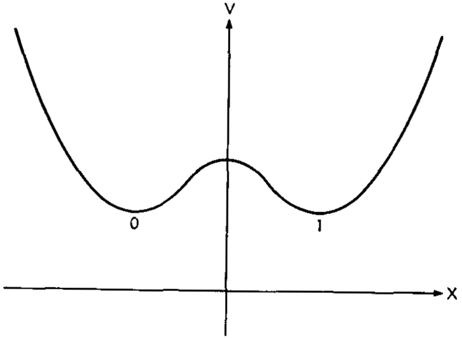
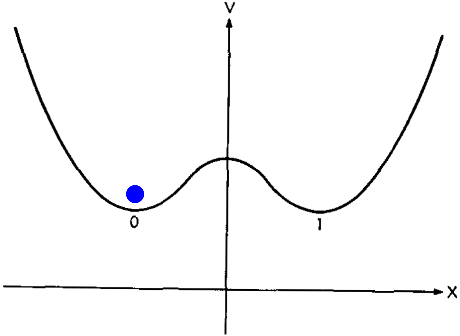


Figure 1 **Bistable potential well.**
x is a generalized coordinate representing quantity which is switched.

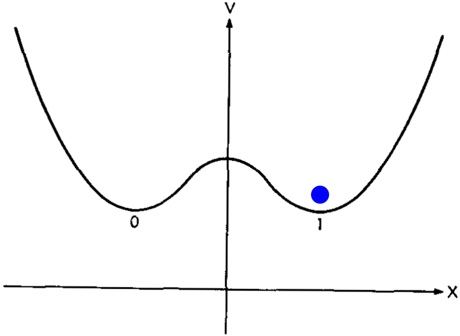
Two states in an analog system. Fix $\rho_f = 0$ as the "erased" state.

$0 \rightarrow 0$



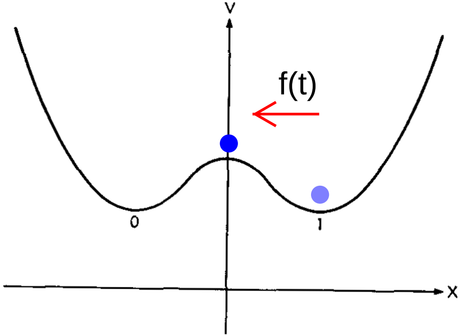
If the system is in 0, we don't have to do anything to erase.

$1 \to 0$



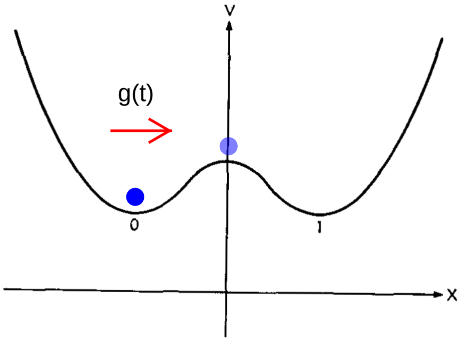
Now consider if the system starts in state 1

$1 \to 0$



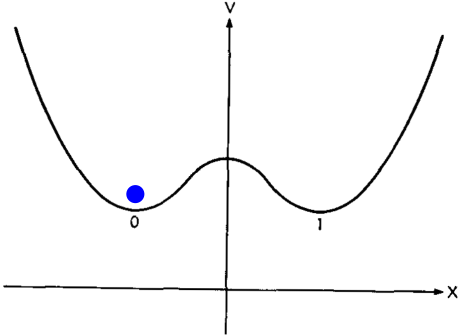
To get to zero, we could apply a force $f(t)$ to the left

$1 \to 0$



Then remove the energy we added with a force $g(t)$ to the right

$1 \to 0$



Resulting in the system "erased" in state zero without any net energy cost.

The catch

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_{i} to the given final state ρ_{f}
- Landauer's motivation (1961): computers operate as a function of circuit connections, not the specific data being handled
- Quantum motivation: we have uncertainty about what state the system is in

The catch

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_i to the given final state ρ_f
- Landauer's motivation (1961): computers operate as a function of circuit connections, not the specific data being handled
- Quantum motivation: we have uncertainty about what state the system is in

The catch

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_{i} to the given final state ρ_{f}
- Landauer's motivation (1961): computers operate as a function of circuit connections, not the specific data being handled
- Quantum motivation: we have uncertainty about what state the system is in

The catch

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_i to the given final state ρ_f
- Landauer's motivation (1961): computers operate as a function of circuit connections, not the specific data being handled
- Quantum motivation: we have uncertainty about what state the system is in

The catch

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_{i} to the given final state ρ_{f}
- Landauer's motivation (1961): computers operate as a function of circuit connections, not the specific data being handled
- Quantum motivation: we have uncertainty about what state the system is in

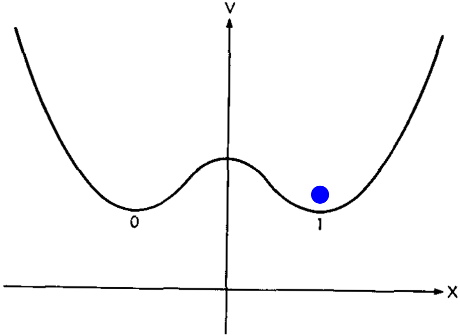
Impossibility

- If we could do a single noiseless operation E that maps $0 \mapsto 0$, $1 \mapsto 0$, we should be able to do the time-reversed process, but inverse is multivalued (0 or 1).
- However, if we allow some noise / heat loss, we cannot perform the time-reversed process. Landauer constructed a method to erase using a damped system.

Impossibility

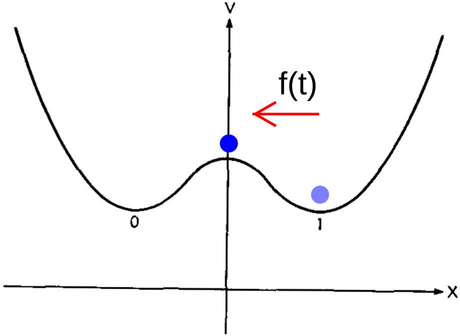
- If we could do a single noiseless operation E that maps $0 \mapsto 0$, $1 \mapsto 0$, we should be able to do the time-reversed process, but inverse is multivalued (0 or 1).
- However, if we allow some noise / heat loss, we cannot perform the time-reversed process. Landauer constructed a method to erase using a damped system.

Landauer's method: Step 1 of 8



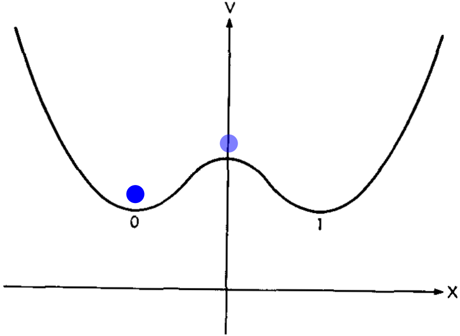
Consider the system starting in the state 1.

Landauer's method: Step 2 of 8



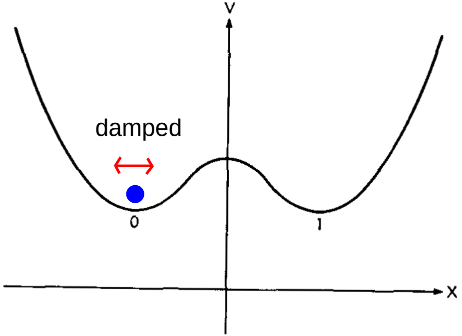
We slowly apply a force $f(t)$ to overcome the potential barrier

Landauer's method: Step 3 of 8



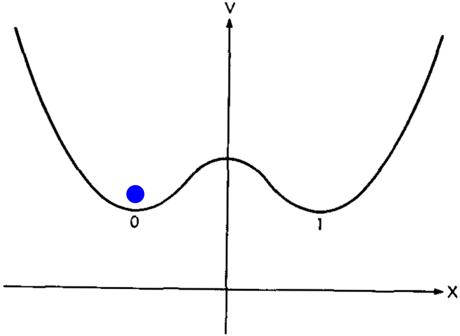
The damped system reduces the kinetic energy enough so the system cannot move back to state 1

Landauer's method: Step 4 of 8



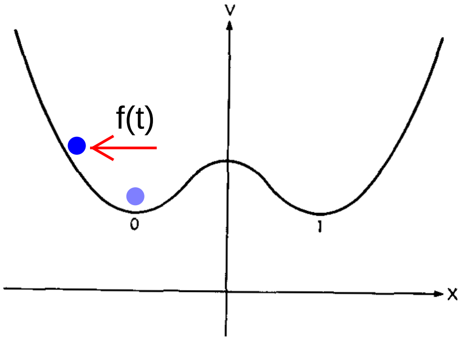
And we have successfully transfered $1 \to 0$

Landauer's method: Step 5 of 8



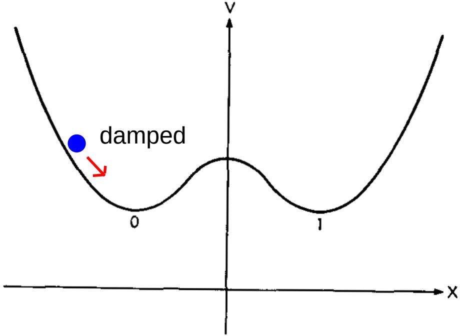
Consider performing the process starting in the state 0.

Landauer's method: Step 6 of 8



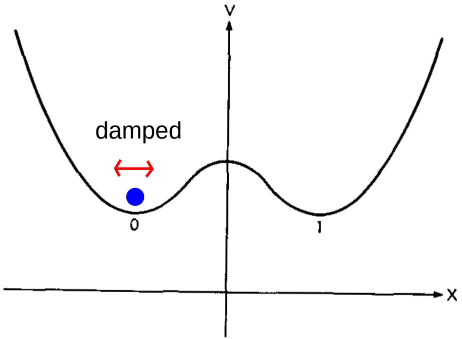
First we add kinetic energy to the system by applying a force to the left.

Landauer's method: Step 7 of 8



But then the damping reduces the kinetic energy enough so that the system cannot cross the potential barrier.

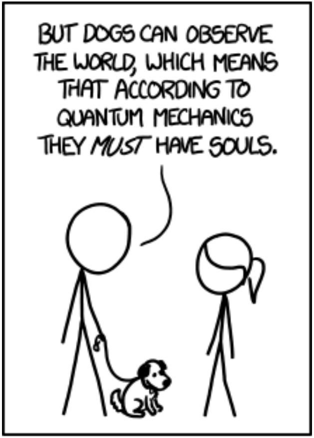
Landauer's method: Step 8 of 8



And the system again settles into the state 0.

Outline

Outline



PROTIP: YOU CAN SAFELY IGNORE ANY SENTENCE THAT INCLUDES THE PHRASE "ACCORDING TO QUANTUM MECHANICS"

http://xkcd.com/1240/

Postulate 1 of 4

Associated to any isolated physical system is a Hilbert space known as the *state space* of the system.

- What is a Hilbert space?
  - Complete inner product space
  - Finite dimensional: $\mathbb{C}^d$, where d is the dimension. Assuming finite dimensional until mentioned otherwise.

The state of the system is completely described by its *density operator*, a positive trace one operator ρ acting on the state space.

- Positive: self-adjoint, all eigenvalues real and non-negative; written $\rho \geq 0$
- Trace one: eigenvalues sum to 1.

If a quantum system is in the state ρ_j with probability p_j then the density operator for the system is $\sum_j p_j \rho_j$.

Postulate 1 of 4

Associated to any isolated physical system is a Hilbert space known as the *state space* of the system.

- What is a Hilbert space?
  - Complete inner product space
  - Finite dimensional: $\mathbb{C}^d$, where d is the dimension. Assuming finite dimensional until mentioned otherwise.

The state of the system is completely described by its *density operator*, a positive trace one operator ρ acting on the state space.

- Positive: self-adjoint, all eigenvalues real and non-negative; written $\rho \geq 0$
- Trace one: eigenvalues sum to 1.

If a quantum system is in the state ρ_j with probability p_j then the density operator for the system is $\sum_j p_j \rho_j$.

Postulate 1 of 4

Associated to any isolated physical system is a Hilbert space known as the *state space* of the system.

- What is a Hilbert space?
  - Complete inner product space
  - Finite dimensional: $\mathbb{C}^d$, where d is the dimension. Assuming finite dimensional until mentioned otherwise.

The state of the system is completely described by its *density operator*, a positive trace one operator ρ acting on the state space.

- Positive: self-adjoint, all eigenvalues real and non-negative; written $\rho \geq 0$
- Trace one: eigenvalues sum to 1.

If a quantum system is in the state ρ_j with probability p_j then the density operator for the system is $\sum_j p_j \rho_j$.

Postulate 1 of 4

Associated to any isolated physical system is a Hilbert space known as the *state space* of the system.

- What is a Hilbert space?
  - Complete inner product space
  - Finite dimensional: $\mathbb{C}^d$, where d is the dimension. Assuming finite dimensional until mentioned otherwise.

The state of the system is completely described by its *density operator*, a positive trace one operator ρ acting on the state space.

- Positive: self-adjoint, all eigenvalues real and non-negative; written $\rho \geq 0$
- Trace one: eigenvalues sum to 1.

If a quantum system is in the state ρ_j with probability p_j then the density operator for the system is $\sum_j p_j \rho_j$.

Postulate 1 of 4

Associated to any isolated physical system is a Hilbert space known as the *state space* of the system.

- What is a Hilbert space?
  - Complete inner product space
  - Finite dimensional: $\mathbb{C}^d$, where d is the dimension. Assuming finite dimensional until mentioned otherwise.

The state of the system is completely described by its *density operator*, a positive trace one operator ρ acting on the state space.

- Positive: self-adjoint, all eigenvalues real and non-negative; written $\rho \geq 0$
- Trace one: eigenvalues sum to 1.

If a quantum system is in the state ρ_j with probability p_j then the density operator for the system is $\sum_j p_j \rho_j$.

Postulate 2 of 4

The evolution of a closed quantum system is given by a unitary transformation. If ρ is the state of the system at time t and ρ' the state of the system at t', then

$$\rho' = U\rho U^*$$

for some unitary U.

- Unitary: $UU^* = U^*U = \mathbb{1}$. Preserves inner product (probability).
- Could be different unitaries between different times; i.e., $U = U(t)$ in general.

Postulate 2 of 4

The evolution of a closed quantum system is given by a unitary transformation. If ρ is the state of the system at time t and ρ' the state of the system at t', then

$$\rho' = U\rho U^*$$

for some unitary U.

- Unitary: $UU^* = U^*U = \mathbb{1}$. Preserves inner product (probability).
- Could be different unitaries between different times; i.e., $U = U(t)$ in general.

Postulate 2 of 4

The evolution of a closed quantum system is given by a unitary transformation. If ρ is the state of the system at time t and ρ' the state of the system at t', then

$$\rho' = U\rho U^*$$

for some unitary U.

- Unitary: $UU^* = U^*U = \mathbb{1}$. Preserves inner product (probability).
- Could be different unitaries between different times; i.e., $U = U(t)$ in general.

Postulate 3 of 4

Quantum measurements are described a collection $\{M_m\}$ of *measurement operators* which act on the state space of the system being measured. The index m refers to the measurement outcomes that may occur. If the state is ρ before the measurement, then the probability that the result m occurs is

$$p(m) = \mathrm{Tr}(M_m^* M_m \rho).$$

After the measurement, the system is in the state

$$\frac{M_m \rho M_m^*}{\mathrm{Tr}(M_m \rho M_m^*)}.$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^* M_m = \mathbb{1}.$$

Postulate 3 of 4

Quantum measurements are described a collection $\{M_m\}$ of *measurement operators* which act on the state space of the system being measured. The index m refers to the measurement outcomes that may occur. If the state is ρ before the measurement, then the probability that the result m occurs is

$$p(m) = \operatorname{Tr}(M_m^* M_m \rho).$$

After the measurement, the system is in the state

$$\frac{M_m \rho M_m^*}{\operatorname{Tr}(M_m \rho M_m^*)}.$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^* M_m = \mathbb{1}.$$

Postulate 3 of 4

Quantum measurements are described a collection $\{M_m\}$ of *measurement operators* which act on the state space of the system being measured. The index m refers to the measurement outcomes that may occur. If the state is ρ before the measurement, then the probability that the result m occurs is

$$p(m) = \mathrm{Tr}(M_m^* M_m \rho).$$

After the measurement, the system is in the state

$$\frac{M_m \rho M_m^*}{\mathrm{Tr}(M_m \rho M_m^*)}.$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^* M_m = \mathbb{1}.$$

Postulate 3 of 4

Quantum measurements are described a collection $\{M_m\}$ of *measurement operators* which act on the state space of the system being measured. The index m refers to the measurement outcomes that may occur. If the state is ρ before the measurement, then the probability that the result m occurs is

$$p(m) = \mathrm{Tr}(M_m^* M_m \rho).$$

After the measurement, the system is in the state

$$\frac{M_m \rho M_m^*}{\mathrm{Tr}(M_m \rho M_m^*)}.$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^* M_m = \mathbb{1}.$$

Observables

A $*$-algebra is a complex vector space $\mathcal{A}$ equipped an associative and distributive multiplication operation, and a map $* : \mathcal{A} \to \mathcal{A}$ that acts like complex conjugation: For any $A, B \in \mathcal{A}$

1. $A^{**} = A$
2. $(AB)^* = B^* A^*$
3. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$.

Given a Hilbert space $\mathcal{H}$, the set $\mathcal{O}_{\mathcal{H}}$ of linear maps $\mathcal{H} \to \mathcal{H}$ is a $*$-algebra. The self-adjoint subset of $\mathcal{O}_{\mathcal{H}}$ is the set of "observables". Notice also that density operators also live in $\mathcal{O}_{H}$.

Observables

A $*$-algebra is a complex vector space $\mathcal{A}$ equipped an associative and distributive multiplication operation, and a map $* : \mathcal{A} \to \mathcal{A}$ that acts like complex conjugation: For any $A, B \in \mathcal{A}$

1. $A^{**} = A$
2. $(AB)^* = B^* A^*$
3. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$.

Given a Hilbert space $\mathcal{H}$, the set $\mathcal{O}_{\mathcal{H}}$ of linear maps $\mathcal{H} \to \mathcal{H}$ is a $*$-algebra. The self-adjoint subset of $\mathcal{O}_{\mathcal{H}}$ is the set of "observables". Notice also that density operators also live in $\mathcal{O}_H$.

Observables

A $*$-algebra is a complex vector space $\mathcal{A}$ equipped an associative and distributive multiplication operation, and a map $* : \mathcal{A} \to \mathcal{A}$ that acts like complex conjugation: For any $A, B \in \mathcal{A}$

1. $A^{**} = A$
2. $(AB)^* = B^* A^*$
3. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$.

Given a Hilbert space $\mathcal{H}$, the set $\mathcal{O}_{\mathcal{H}}$ of linear maps $\mathcal{H} \to \mathcal{H}$ is a $*$-algebra. The self-adjoint subset of $\mathcal{O}_{\mathcal{H}}$ is the set of "observables". Notice also that density operators also live in $\mathcal{O}_H$.

Remark on postulate 3

If we consider a physical observable O then O is a self-adjoint operator. We can write

$$O = \sum_i \lambda_i O_i = \sum_i \lambda_i O_i^2 = \sum_i \lambda_i O_i^* O_i$$

where λ_i are the eigenvalues of O and O_i the orthogonal projections into the eigenspaces. Then if the system is in the state ρ,

$$\mathrm{Tr}(O\rho) = \sum_i \lambda_i \mathrm{Tr}(O_i^* O_i \rho) = \sum_i \lambda_i p(i) = \mathbb{E}(O)$$

the expectation value of O.

Remark on postulate 3

If we consider a physical observable O then O is a self-adjoint operator. We can write

$$O = \sum_i \lambda_i O_i = \sum_i \lambda_i O_i^2 = \sum_i \lambda_i O_i^* O_i$$

where λ_i are the eigenvalues of O and O_i the orthogonal projections into the eigenspaces. Then if the system is in the state ρ,

$$\mathrm{Tr}(O\rho) = \sum_i \lambda_i \mathrm{Tr}(O_i^* O_i \rho) = \sum_i \lambda_i p(i) = \mathbb{E}(O)$$

the expectation value of O.

Remark on postulate 3

If we consider a physical observable O then O is a self-adjoint operator. We can write

$$O = \sum_i \lambda_i O_i = \sum_i \lambda_i O_i^2 = \sum_i \lambda_i O_i^* O_i$$

where λ_i are the eigenvalues of O and O_i the orthogonal projections into the eigenspaces. Then if the system is in the state ρ,

$$\mathrm{Tr}(O\rho) = \sum_i \lambda_i \,\mathrm{Tr}(O_i^* O_i \rho) = \sum_i \lambda_i p(i) = \mathbb{E}(O)$$

the expectation value of O.

Hamiltonian

- For a physical finite-dimensional system with Hilbert space $\mathcal{H}$, the energy of the system is often described by an observable (self-adjoint operator acting on $\mathcal{H}$) called the Hamiltonian, denoted by H.
  - The eigenvalues of H are then the possible measured values of the energy of the system.
- The Hamiltonian also generates the time evolution unitary U from postulate 2. U satisfies differential equation

$$i\frac{\mathrm{d}U}{\mathrm{d}t} = HU$$

Hamiltonian

- For a physical finite-dimensional system with Hilbert space $\mathcal{H}$, the energy of the system is often described by an observable (self-adjoint operator acting on $\mathcal{H}$) called the Hamiltonian, denoted by H.
  - The eigenvalues of H are then the possible measured values of the energy of the system.
- The Hamiltonian also generates the time evolution unitary U from postulate 2. U satisfies differential equation

$$i\frac{\mathrm{d}U}{\mathrm{d}t} = HU$$

Hamiltonian

- For a physical finite-dimensional system with Hilbert space $\mathcal{H}$, the energy of the system is often described by an observable (self-adjoint operator acting on $\mathcal{H}$) called the Hamiltonian, denoted by H.
  - The eigenvalues of H are then the possible measured values of the energy of the system.
- The Hamiltonian also generates the time evolution unitary U from postulate 2. U satisfies differential equation

$$i\frac{\mathrm{d}U}{\mathrm{d}t} = HU$$

Postulate 4 of 4

The state space of the complete physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n$.

Tensor products

- What is the tensor product? Consider two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases $\{a_i\}$ and $\{b_j\}$
- Tensor product of vectors: We can think of $a_i \otimes b_j$ as formal symbols, and define the tensor product of other vectors $\psi = \sum_i \alpha_i a_i \in \mathcal{H}_A$, $\varphi = \sum_j \beta_j b_j \in \mathcal{H}_B$ ($\alpha_i, \beta_j \in \mathbb{C}$) by imposing linearity: $\psi \otimes \varphi = \sum_{ij} \alpha_i \beta_j a_i \otimes b_j$.
- Of spaces: we define $\mathcal{H}_A \otimes \mathcal{H}_B := \operatorname{span}(\{a_i \otimes b_j\})$, with the inner product $\langle a_i \otimes b_j, a_{i'} \otimes b_{j'} \rangle = \delta_{ii'}\delta_{jj'}$.
- Of operators: If X acts on $\mathcal{H}_A$ ($X \in \mathcal{O}_{\mathcal{H}_A}$) and Y on $\mathcal{H}_B$, then $X \otimes Y$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ by $(X \otimes Y)\psi \otimes \varphi = X\psi \otimes Y\varphi$. Then, $\mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}$ is the $*$-algebra generated by such operators.

Tensor products

- What is the tensor product? Consider two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases $\{a_i\}$ and $\{b_j\}$
- Tensor product of vectors: We can think of $a_i \otimes b_j$ as formal symbols, and define the tensor product of other vectors $\psi = \sum_i \alpha_i a_i \in \mathcal{H}_A$, $\varphi = \sum_j \beta_j b_j \in \mathcal{H}_B$ ($\alpha_i, \beta_j \in \mathbb{C}$) by imposing linearity: $\psi \otimes \varphi = \sum_{ij} \alpha_i \beta_j a_i \otimes b_j$.
- Of spaces: we define $\mathcal{H}_A \otimes \mathcal{H}_B := \operatorname{span}(\{a_i \otimes b_j\})$, with the inner product $\langle a_i \otimes b_j, a_{i'} \otimes b_{j'} \rangle = \delta_{ii'}\delta_{jj'}$.
- Of operators: If X acts on $\mathcal{H}_A$ ($X \in \mathcal{O}_{\mathcal{H}_A}$) and Y on $\mathcal{H}_B$, then $X \otimes Y$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ by $(X \otimes Y)\psi \otimes \varphi = X\psi \otimes Y\varphi$. Then, $\mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}$ is the $*$-algebra generated by such operators.

Tensor products

- What is the tensor product? Consider two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases $\{a_i\}$ and $\{b_j\}$
- Tensor product of vectors: We can think of $a_i \otimes b_j$ as formal symbols, and define the tensor product of other vectors $\psi = \sum_i \alpha_i a_i \in \mathcal{H}_A$, $\varphi = \sum_j \beta_j b_j \in \mathcal{H}_B$ $(\alpha_i, \beta_j \in \mathbb{C})$ by imposing linearity: $\psi \otimes \varphi = \sum_{ij} \alpha_i \beta_j a_i \otimes b_j$.
- Of spaces: we define $\mathcal{H}_A \otimes \mathcal{H}_B := \operatorname{span}(\{a_i \otimes b_j\})$, with the inner product $\langle a_i \otimes b_j, a_{i'} \otimes b_{j'} \rangle = \delta_{ii'} \delta_{jj'}$.
- Of operators: If X acts on $\mathcal{H}_A$ $(X \in \mathcal{O}_{\mathcal{H}_A})$ and Y on $\mathcal{H}_B$, then $X \otimes Y$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ by $(X \otimes Y)\psi \otimes \varphi = X\psi \otimes Y\varphi$. Then, $\mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}$ is the $*$-algebra generated by such operators.

Tensor products

- What is the tensor product? Consider two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases $\{a_i\}$ and $\{b_j\}$
- Tensor product of vectors: We can think of $a_i \otimes b_j$ as formal symbols, and define the tensor product of other vectors $\psi = \sum_i \alpha_i a_i \in \mathcal{H}_A$, $\varphi = \sum_j \beta_j b_j \in \mathcal{H}_B$ ($\alpha_i, \beta_j \in \mathbb{C}$) by imposing linearity: $\psi \otimes \varphi = \sum_{ij} \alpha_i \beta_j a_i \otimes b_j$.
- Of spaces: we define $\mathcal{H}_A \otimes \mathcal{H}_B := \operatorname{span}(\{a_i \otimes b_j\})$, with the inner product $\langle a_i \otimes b_j, a_{i'} \otimes b_{j'} \rangle = \delta_{ii'}\delta_{jj'}$.
- Of operators: If X acts on $\mathcal{H}_A$ ($X \in \mathcal{O}_{\mathcal{H}_A}$) and Y on $\mathcal{H}_B$, then $X \otimes Y$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ by $(X \otimes Y)\psi \otimes \varphi = X\psi \otimes Y\varphi$. Then, $\mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}$ is the $*$-algebra generated by such operators.

Remark on Postulate 4

- Given a composite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and state ρ^{AB} a density operator on the joint Hilbert space, we would like a way to recover the state on system A, which we'll denote ρ^A.
- We would like to have that if we have an observable $O \in \mathcal{O}_{\mathcal{H}_A}$ and any density operator ρ^{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, the value we measure for O only depends on ρ^A.

Remark on Postulate 4

- Given a composite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and state ρ^{AB} a density operator on the joint Hilbert space, we would like a way to recover the state on system A, which we'll denote ρ^A.
- We would like to have that if we have an observable $O \in \mathcal{O}_{\mathcal{H}_A}$ and any density operator ρ^{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, the value we measure for O only depends on ρ^A.

The partial trace: First definition

Theorem

Consider two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.
Then there is a unique map $\Phi : \mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B} \to \mathcal{O}_{\mathcal{H}_A}$ such that for all $C \in \mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}$ and $B \in \mathcal{O}_{\mathcal{H}_A}$,

$$\mathrm{Tr}\big(\underbrace{(B \otimes \mathbb{1})C}_{\in \mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}}\big) = \mathrm{Tr}\big(\underbrace{B\Phi(C)}_{\in \mathcal{O}_{\mathcal{H}_A}}\big)$$

We call Φ the *partial trace over B*, and denote it by Tr_B.

- Interpretation: $C = \rho^{AB}$, $\Phi(C) = \mathrm{Tr}_B C =: \rho^A$. If we only measure B on the joint system (so measurement on both systems is $B \otimes \mathbb{1}$), we get the same as just measuring ρ^A on the A system.

The partial trace: First definition

Theorem

Consider two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.
Then there is a unique map $\Phi : \mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B} \to \mathcal{O}_{\mathcal{H}_A}$ such that for all $C \in \mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}$ and $B \in \mathcal{O}_{\mathcal{H}_A}$,

$$\mathrm{Tr}(\underbrace{(B \otimes \mathbb{1})C}_{\in \mathcal{O}_{\mathcal{H}_A} \otimes \mathcal{O}_{\mathcal{H}_B}}) = \mathrm{Tr}(\underbrace{B\Phi(C)}_{\in \mathcal{O}_{\mathcal{H}_A}})$$

We call Φ the *partial trace over B*, and denote it by Tr_B.

- Interpretation: $C = \rho^{AB}$, $\Phi(C) = \mathrm{Tr}_B\, C =: \rho^A$. If we only measure B on the joint system (so measurement on both systems is $B \otimes \mathbb{1}$), we get the same as just measuring ρ^A on the A system.

Operational definition of partial trace

- We can always write our state as $\rho^{AB} = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B$ for some numbers λ_i and $\rho_i^A \in \mathcal{O}_{\mathcal{H}_A}$, $\rho_i^B \in \mathcal{O}_{\mathcal{H}_B}$.
- We define the *reduced density operator* on system A by

$$\rho^A := \operatorname{Tr}_B \rho^{AB} = \sum_i \lambda_i \operatorname{Tr}_B(\rho_i^A \otimes \rho_i^B) = \sum_i \lambda_i \underbrace{\operatorname{Tr}(\rho_i^B)}_{\in\mathbb{C}} \rho_i^A$$

Operational definition of partial trace

- We can always write our state as $\rho^{AB} = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B$ for some numbers λ_i and $\rho_i^A \in \mathcal{O}_{\mathcal{H}_A}$, $\rho_i^B \in \mathcal{O}_{\mathcal{H}_B}$.
- We define the *reduced density operator* on system A by

$$\rho^A := \operatorname{Tr}_B \rho^{AB} = \sum_i \lambda_i \operatorname{Tr}_B(\rho_i^A \otimes \rho_i^B) = \sum_i \lambda_i \underbrace{\operatorname{Tr}(\rho_i^B)}_{\in\mathbb{C}} \rho_i^A$$

Operational definition of partial trace

- We can always write our state as $\rho^{AB} = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B$ for some numbers λ_i and $\rho_i^A \in \mathcal{O}_{\mathcal{H}_A}$, $\rho_i^B \in \mathcal{O}_{\mathcal{H}_B}$.
- We define the *reduced density operator* on system A by

$$\rho^A := \operatorname{Tr}_B \rho^{AB} = \sum_i \lambda_i \operatorname{Tr}_B(\rho_i^A \otimes \rho_i^B) = \sum_i \lambda_i \underbrace{\operatorname{Tr}(\rho_i^B)}_{\in \mathbb{C}} \rho_i^A$$

Operational definition of partial trace

- We can always write our state as $\rho^{AB} = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B$ for some numbers λ_i and $\rho_i^A \in \mathcal{O}_{\mathcal{H}_A}$, $\rho_i^B \in \mathcal{O}_{\mathcal{H}_B}$.
- We define the *reduced density operator* on system A by

$$\rho^A := \operatorname{Tr}_B \rho^{AB} = \sum_i \lambda_i \operatorname{Tr}_B(\rho_i^A \otimes \rho_i^B) = \sum_i \lambda_i \underbrace{\operatorname{Tr}(\rho_i^B)}_{\in\mathbb{C}} \rho_i^A$$

Outline

Von Neumann Entropy

- We consider a density operator ρ with eigenvalues $\{\lambda_i\}$. Then we define the entropy of ρ as

$$S(\rho) := -\operatorname{Tr} \rho \log \rho = -\sum_i \lambda_i \log \lambda_i$$

- $S(\rho) \geq 0$ with equality iff ρ is a projector. Why zero for projector? No uncertainty; exactly one non-zero eigenvalue.
- $S(\rho) \leq \log d$ with equality iff $\rho = \mathbb{1}/d$, the "chaotic state". Maximal uncertainty; every outcome equally likely.
- Since $S(\rho)$ only depends on the eigenvalues of ρ, for any unitary U, $S(\rho) = S(U\rho U^*)$.
- Furthermore, for two density operators ρ_1, ρ_2, we have $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$.

Von Neumann Entropy

- We consider a density operator ρ with eigenvalues $\{\lambda_i\}$. Then we define the entropy of ρ as

$$S(\rho) := -\operatorname{Tr}\rho\log\rho = -\sum_i \lambda_i \log\lambda_i$$

- $S(\rho) \geq 0$ with equality iff ρ is a projector. Why zero for projector? No uncertainty; exactly one non-zero eigenvalue.
- $S(\rho) \leq \log d$ with equality iff $\rho = \mathbb{1}/d$, the "chaotic state". Maximal uncertainty; every outcome equally likely.
- Since $S(\rho)$ only depends on the eigenvalues of ρ, for any unitary U, $S(\rho) = S(U\rho U^*)$.
- Furthermore, for two density operators ρ_1, ρ_2, we have $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$.

Von Neumann Entropy

- We consider a density operator ρ with eigenvalues $\{\lambda_i\}$. Then we define the entropy of ρ as

$$S(\rho) := -\operatorname{Tr} \rho \log \rho = -\sum_i \lambda_i \log \lambda_i$$

- $S(\rho) \geq 0$ with equality iff ρ is a projector. Why zero for projector? No uncertainty; exactly one non-zero eigenvalue.
- $S(\rho) \leq \log d$ with equality iff $\rho = \mathbb{1}/d$, the "chaotic state". Maximal uncertainty; every outcome equally likely.
- Since $S(\rho)$ only depends on the eigenvalues of ρ, for any unitary U, $S(\rho) = S(U\rho U^*)$.
- Furthermore, for two density operators ρ_1, ρ_2, we have $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$.

Von Neumann Entropy

- We consider a density operator ρ with eigenvalues $\{\lambda_i\}$. Then we define the entropy of ρ as

$$S(\rho) := -\operatorname{Tr} \rho \log \rho = -\sum_i \lambda_i \log \lambda_i$$

- $S(\rho) \geq 0$ with equality iff ρ is a projector. Why zero for projector? No uncertainty; exactly one non-zero eigenvalue.
- $S(\rho) \leq \log d$ with equality iff $\rho = \mathbb{1}/d$, the "chaotic state". Maximal uncertainty; every outcome equally likely.
- Since $S(\rho)$ only depends on the eigenvalues of ρ, for any unitary U, $S(\rho) = S(U\rho U^*)$.
- Furthermore, for two density operators ρ_1, ρ_2, we have $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$.

Von Neumann Entropy

- We consider a density operator ρ with eigenvalues $\{\lambda_i\}$. Then we define the entropy of ρ as

$$S(\rho) := -\operatorname{Tr} \rho \log \rho = -\sum_i \lambda_i \log \lambda_i$$

- $S(\rho) \geq 0$ with equality iff ρ is a projector. Why zero for projector? No uncertainty; exactly one non-zero eigenvalue.
- $S(\rho) \leq \log d$ with equality iff $\rho = \mathbb{1}/d$, the "chaotic state". Maximal uncertainty; every outcome equally likely.
- Since $S(\rho)$ only depends on the eigenvalues of ρ, for any unitary U, $S(\rho) = S(U\rho U^*)$.
- Furthermore, for two density operators ρ_1, ρ_2, we have $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$.

Relative entropy

For two states ρ_1, ρ_2, we define the relative entropy as

$$S(\rho_1|\rho_2) := \mathrm{Tr}(\rho_1(\log \rho_1 - \log \rho_2))$$

Furthermore, $S(\rho_1|\rho_2) \geq 0$ with equality iff $\rho_1 = \rho_2$.

Relative entropy

For two states ρ_1, ρ_2, we define the relative entropy as

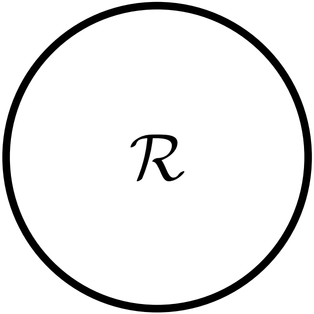
$$S(\rho_1|\rho_2) := \mathrm{Tr}(\rho_1(\log \rho_1 - \log \rho_2))$$

Furthermore, $S(\rho_1|\rho_2) \geq 0$ with equality iff $\rho_1 = \rho_2$.

Outline

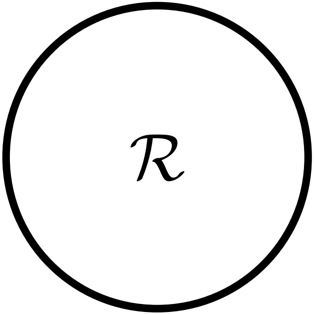
Setup

Reservoir



Setup

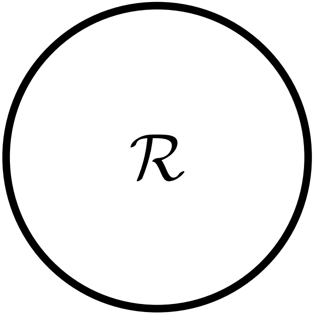
Reservoir



$\mathcal{H}_\mathcal{R}$, $\dim \mathcal{H}_\mathcal{R} < \infty$

Setup

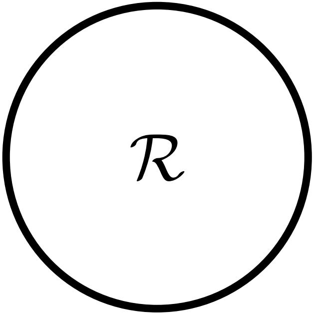
Reservoir



$\mathcal{H}_{\mathcal{R}}$, $\dim \mathcal{H}_{\mathcal{R}} < \infty$
Hamiltonian $H_{\mathcal{R}}$

Setup

Reservoir

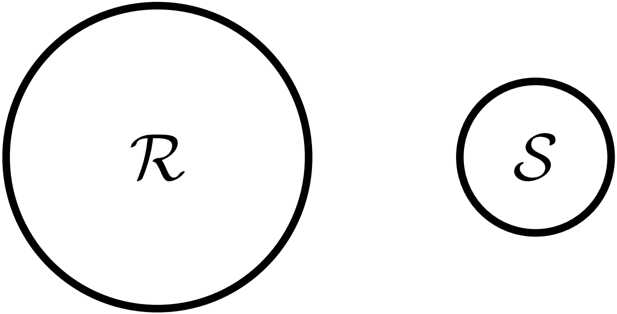


$\mathcal{H}_{\mathcal{R}}$, $\dim \mathcal{H}_{\mathcal{R}} < \infty$
Hamiltonian $H_{\mathcal{R}}$
State ν_{i} on $\mathcal{H}_{\mathcal{R}}$

Setup

Reservoir

System



$\mathcal{H}_{\mathcal{R}}$, $\dim \mathcal{H}_{\mathcal{R}} < \infty$

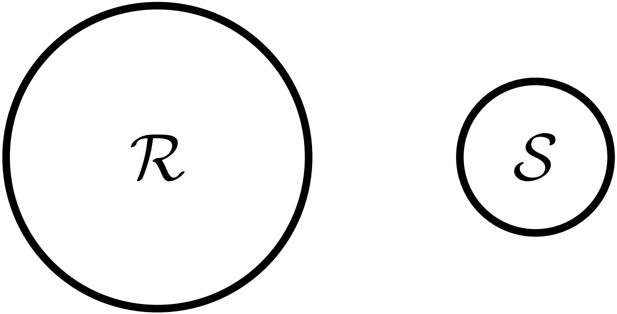
Hamiltonian $H_{\mathcal{R}}$

State ν_{i} on $\mathcal{H}_{\mathcal{R}}$

Setup

Reservoir

System

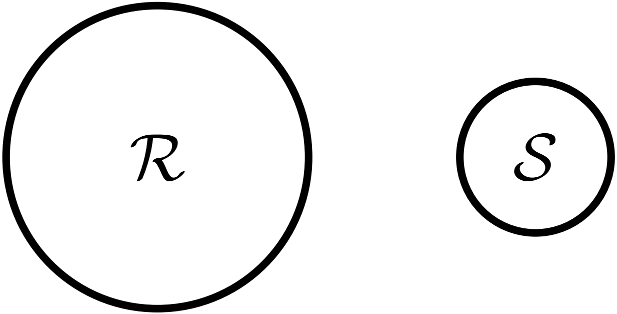


$\mathcal{H}_\mathcal{R}$, $\dim \mathcal{H}_\mathcal{R} < \infty$
Hamiltonian $H_\mathcal{R}$
State ν_i on $\mathcal{H}_\mathcal{R}$

$\mathcal{H}_\mathcal{S}$, $\dim \mathcal{H}_\mathcal{S} = d < \infty$

Setup

Reservoir | System



$\mathcal{H}_{\mathcal{R}}$, $\dim \mathcal{H}_{\mathcal{R}} < \infty$
Hamiltonian $H_{\mathcal{R}}$
State ν_{i} on $\mathcal{H}_{\mathcal{R}}$

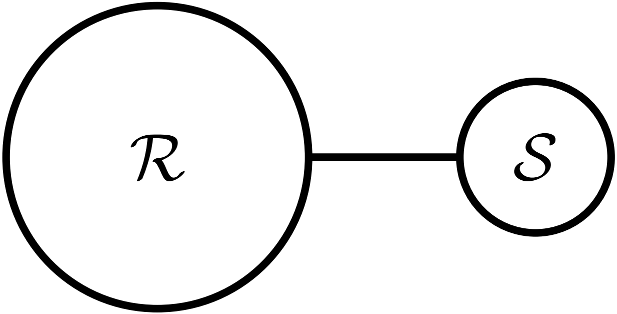
$\mathcal{H}_{\mathcal{S}}$, $\dim \mathcal{H}_{\mathcal{S}} = d < \infty$

State ρ_{i} on $\mathcal{H}_{\mathcal{S}}$

Setup

Reservoir

System



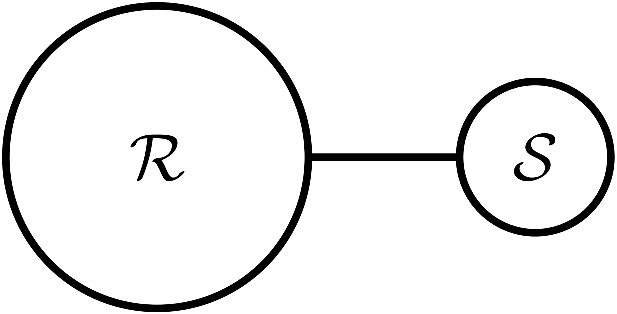
$\mathcal{H}_\mathcal{R}$, $\dim \mathcal{H}_\mathcal{R} < \infty$
Hamiltonian $H_\mathcal{R}$
State ν_i on $\mathcal{H}_\mathcal{R}$

$\mathcal{H}_\mathcal{S}$, $\dim \mathcal{H}_\mathcal{S} = d < \infty$

State ρ_i on $\mathcal{H}_\mathcal{S}$

Setup

Reservoir | System



$\mathcal{H}_{\mathcal{R}}$, $\dim \mathcal{H}_{\mathcal{R}} < \infty$
Hamiltonian $H_{\mathcal{R}}$
State ν_{i} on $\mathcal{H}_{\mathcal{R}}$

$\mathcal{H}_{\mathcal{S}}$, $\dim \mathcal{H}_{\mathcal{S}} = d < \infty$
State ρ_{i} on $\mathcal{H}_{\mathcal{S}}$

Joint state: $\omega_{\mathsf{i}} = \rho_{\mathsf{i}} \otimes \nu_{\mathsf{i}}$

The state on the reservoir, ν_{i}

- We fix the initial state on the reservoir to be $\nu_{\mathrm{i}} = e^{-\beta H_{\mathcal{R}}}/Z =: \rho_\beta$, where $Z = \mathrm{Tr}\left(e^{-\beta H_{\mathcal{R}}}\right)$, and $\beta = 1/T$ (T being the temperature).
- This is known as a Gibbs state or thermal equilibrium state.

Theorem

For a Hamiltonian H and energy $E \in \mathbb{R}$, there exists a $\beta \in \mathbb{R}$ such that $\mathrm{Tr}(\rho_\beta H) = E$. Moreover, if $\mathfrak{S}_E = \{\rho \text{ is a state such that } \mathrm{Tr}(\rho H) = E\}$, then

$$\max_{\rho \in \mathfrak{S}_E} S(\rho) = S(\rho_\beta)$$

that is, ρ_β maximizes entropy at fixed energy.

The state on the reservoir, ν_{i}

- We fix the initial state on the reservoir to be $\nu_{\mathrm{i}} = e^{-\beta H_{\mathcal{R}}}/Z =: \rho_\beta$, where $Z = \mathrm{Tr}\left(e^{-\beta H_{\mathcal{R}}}\right)$, and $\beta = 1/T$ (T being the temperature).
- This is known as a Gibbs state or thermal equilibrium state.

Theorem

For a Hamiltonian H and energy $E \in \mathbb{R}$, there exists a $\beta \in \mathbb{R}$ such that $\mathrm{Tr}(\rho_\beta H) = E$. Moreover, if $\mathfrak{S}_E = \{\rho \text{ is a state such that } \mathrm{Tr}(\rho H) = E\}$, then

$$\max_{\rho \in \mathfrak{S}_E} S(\rho) = S(\rho_\beta)$$

that is, ρ_β maximizes entropy at fixed energy.

The state on the reservoir, ν_i

- We fix the initial state on the reservoir to be $\nu_\mathrm{i} = e^{-\beta H_\mathcal{R}}/Z =: \rho_\beta$, where $Z = \mathrm{Tr}\left(e^{-\beta H_\mathcal{R}}\right)$, and $\beta = 1/T$ (T being the temperature).
- This is known as a Gibbs state or thermal equilibrium state.

Theorem

For a Hamiltonian H and energy $E \in \mathbb{R}$, there exists a $\beta \in \mathbb{R}$ such that $\mathrm{Tr}(\rho_\beta H) = E$. Moreover, if $\mathfrak{S}_E = \{\rho \text{ is a state such that } \mathrm{Tr}(\rho H) = E\}$, then

$$\max_{\rho \in \mathfrak{S}_E} S(\rho) = S(\rho_\beta)$$

that is, ρ_β maximizes entropy at fixed energy.

The state on the reservoir, ν_{i}

- We fix the initial state on the reservoir to be $\nu_{\mathrm{i}} = e^{-\beta H_{\mathcal{R}}}/Z =: \rho_\beta$, where $Z = \operatorname{Tr}\left(e^{-\beta H_{\mathcal{R}}}\right)$, and $\beta = 1/T$ (T being the temperature).
- This is known as a Gibbs state or thermal equilibrium state.

Theorem

For a Hamiltonian H and energy $E \in \mathbb{R}$, there exists a $\beta \in \mathbb{R}$ such that $\operatorname{Tr}(\rho_\beta H) = E$. Moreover, if $\mathfrak{S}_E = \{\rho \text{ is a state such that } \operatorname{Tr}(\rho H) = E\}$, then

$$\max_{\rho \in \mathfrak{S}_E} S(\rho) = S(\rho_\beta)$$

that is, ρ_β maximizes entropy at fixed energy.

The state on the reservoir, ν_i

- We fix the initial state on the reservoir to be $\nu_\mathrm{i} = e^{-\beta H_\mathcal{R}}/Z =: \rho_\beta$, where $Z = \mathrm{Tr}\left(e^{-\beta H_\mathcal{R}}\right)$, and $\beta = 1/T$ (T being the temperature).
- This is known as a Gibbs state or thermal equilibrium state.

Theorem

For a Hamiltonian H and energy $E \in \mathbb{R}$, there exists a $\beta \in \mathbb{R}$ such that $\mathrm{Tr}(\rho_\beta H) = E$. Moreover, if $\mathfrak{S}_E = \{\rho \text{ is a state such that } \mathrm{Tr}(\rho H) = E\}$, then

$$\max_{\rho \in \mathfrak{S}_E} S(\rho) = S(\rho_\beta)$$

that is, ρ_β maximizes entropy at fixed energy.

Dynamics

- We are considering processes $\rho_{\mathsf{i}} \to \rho_{\mathsf{f}}$. Since we are working in quantum mechanics, we consider operations unitary on the *joint* system $U : \mathcal{H} \to \mathcal{H}$
- This induces a transformation on the joint state: $\omega_U = U \omega_{\mathsf{i}} U^*$
  - The transformed state on the system $\mathcal{S}$ is given by $\rho_U = \mathrm{Tr}_{\mathcal{R}}(\omega_U)$. We'd like $\rho_{\mathsf{f}} = \rho_U$, i.e. U is the operation that induces $\rho_{\mathsf{i}} \to \rho_{\mathsf{f}}$.
  - On the reservoir $\mathcal{R}$, we have $\nu_U = \mathrm{Tr}_{\mathcal{S}}(\omega_U)$

Dynamics

- We are considering processes $\rho_\mathsf{i} \to \rho_\mathsf{f}$. Since we are working in quantum mechanics, we consider operations unitary on the *joint* system $U : \mathcal{H} \to \mathcal{H}$
- This induces a transformation on the joint state: $\omega_U = U\omega_\mathsf{i} U^*$
  - The transformed state on the system $\mathcal{S}$ is given by $\rho_U = \mathrm{Tr}_\mathcal{R}(\omega_U)$. We'd like $\rho_\mathsf{f} = \rho_U$, i.e. U is the operation that induces $\rho_\mathsf{i} \to \rho_\mathsf{f}$.
  - On the reservoir $\mathcal{R}$, we have $\nu_U = \mathrm{Tr}_\mathcal{S}(\omega_U)$

Dynamics

- We are considering processes $\rho_{\mathsf{i}} \to \rho_{\mathsf{f}}$. Since we are working in quantum mechanics, we consider operations unitary on the *joint* system $U : \mathcal{H} \to \mathcal{H}$
- This induces a transformation on the joint state: $\omega_U = U\omega_{\mathsf{i}}U^*$
  - The transformed state on the system $\mathcal{S}$ is given by $\rho_U = \mathrm{Tr}_{\mathcal{R}}(\omega_U)$. We'd like $\rho_{\mathsf{f}} = \rho_U$, i.e. U is the operation that induces $\rho_{\mathsf{i}} \to \rho_{\mathsf{f}}$.
  - On the reservoir $\mathcal{R}$, we have $\nu_U = \mathrm{Tr}_{\mathcal{S}}(\omega_U)$

Dynamics

- We are considering processes $\rho_{\mathsf{i}} \to \rho_{\mathsf{f}}$. Since we are working in quantum mechanics, we consider operations unitary on the *joint* system $U : \mathcal{H} \to \mathcal{H}$
- This induces a transformation on the joint state: $\omega_U = U\omega_{\mathsf{i}}U^*$
    - The transformed state on the system $\mathcal{S}$ is given by $\rho_U = \mathrm{Tr}_{\mathcal{R}}(\omega_U)$. We'd like $\rho_{\mathsf{f}} = \rho_U$, i.e. U is the operation that induces $\rho_{\mathsf{i}} \to \rho_{\mathsf{f}}$.
    - On the reservoir $\mathcal{R}$, we have $\nu_U = \mathrm{Tr}_{\mathcal{S}}(\omega_U)$

Quantities to consider

- Energy (heat) loss from the system is the energy gained from the reservoir (called ΔQ)
  - Initial energy of the reservoir is the expectation of the Hamiltonian: $\mathrm{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}})$.
  - Final energy is $\mathrm{Tr}(\nu_U H_{\mathcal{R}})$.
- Entropy change of the system is $\Delta S = S(\rho_U) - S(\rho_{\mathrm{i}})$.
- Landauer's bound: $\beta \Delta Q \geq \Delta S$.
  - If ρ_U is a pure state and $\rho_{\mathrm{i}} = \mathbb{1}/d$, $\Delta S = \log d$, and Landauer's bound reads $\Delta Q \geq T \log d$.

Quantities to consider

- Energy (heat) loss from the system is the energy gained from the reservoir (called ΔQ)
  - Initial energy of the reservoir is the expectation of the Hamiltonian: $\mathrm{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}})$.
  - Final energy is $\mathrm{Tr}(\nu_U H_{\mathcal{R}})$.
- Entropy change of the system is $\Delta S = S(\rho_U) - S(\rho_{\mathrm{i}})$.
- Landauer's bound: $\beta \Delta Q \geq \Delta S$.
  - If ρ_U is a pure state and $\rho_{\mathrm{i}} = \mathbb{1}/d$, $\Delta S = \log d$, and Landauer's bound reads $\Delta Q \geq T \log d$.

Quantities to consider

- Energy (heat) loss from the system is the energy gained from the reservoir (called ΔQ)
  - Initial energy of the reservoir is the expectation of the Hamiltonian: $\mathrm{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}})$.
  - Final energy is $\mathrm{Tr}(\nu_U H_{\mathcal{R}})$.
- Entropy change of the system is $\Delta S = S(\rho_U) - S(\rho_{\mathrm{i}})$.
- Landauer's bound: $\beta \Delta Q \geq \Delta S$.
  - If ρ_U is a pure state and $\rho_{\mathrm{i}} = \mathbb{1}/d$, $\Delta S = \log d$, and Landauer's bound reads $\Delta Q \geq T \log d$.

Quantities to consider

- Energy (heat) loss from the system is the energy gained from the reservoir (called ΔQ)
  - Initial energy of the reservoir is the expectation of the Hamiltonian: $\mathrm{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}})$.
  - Final energy is $\mathrm{Tr}(\nu_U H_{\mathcal{R}})$.
- Entropy change of the system is $\Delta S = S(\rho_U) - S(\rho_{\mathrm{i}})$.
- Landauer's bound: $\beta \Delta Q \geq \Delta S$.
  - If ρ_U is a pure state and $\rho_{\mathrm{i}} = \mathbb{1}/d$, $\Delta S = \log d$, and Landauer's bound reads $\Delta Q \geq T \log d$.

Quantities to consider

- Energy (heat) loss from the system is the energy gained from the reservoir (called ΔQ)
  - Initial energy of the reservoir is the expectation of the Hamiltonian: $\mathrm{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}})$.
  - Final energy is $\mathrm{Tr}(\nu_U H_{\mathcal{R}})$.
- Entropy change of the system is $\Delta S = S(\rho_U) - S(\rho_{\mathrm{i}})$.
- Landauer's bound: $\beta \Delta Q \geq \Delta S$.
  - If ρ_U is a pure state and $\rho_{\mathrm{i}} = \mathbb{1}/d$, $\Delta S = \log d$, and Landauer's bound reads $\Delta Q \geq T \log d$.

Quantities to consider

- Energy (heat) loss from the system is the energy gained from the reservoir (called ΔQ)
  - Initial energy of the reservoir is the expectation of the Hamiltonian: $\mathrm{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}})$.
  - Final energy is $\mathrm{Tr}(\nu_U H_{\mathcal{R}})$.
- Entropy change of the system is $\Delta S = S(\rho_U) - S(\rho_{\mathrm{i}})$.
- Landauer's bound: $\beta \Delta Q \geq \Delta S$.
  - If ρ_U is a pure state and $\rho_{\mathrm{i}} = \mathbb{1}/d$, $\Delta S = \log d$, and Landauer's bound reads $\Delta Q \geq T \log d$.

Calculations (1/2)

First,

$$S(\nu_{\mathrm{i}}) := -\operatorname{Tr}(\nu_{\mathrm{i}} \log \nu_{\mathrm{i}}) = -\operatorname{Tr}\left(\nu_{\mathrm{i}} \log\left(\frac{e^{-\beta H_{\mathcal{R}}}}{Z}\right)\right)$$

$$= -\operatorname{Tr}\left(\nu_{\mathrm{i}} \log\left(e^{-\beta H_{\mathcal{R}}}\right)\right) + \log Z \cdot \cancelto{1}{\operatorname{Tr}(\nu_{\mathrm{i}})}$$

$$= \beta \operatorname{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}}) + \log Z$$

Next,

$$S(\omega_U) = S(\omega_{\mathrm{i}}) = S(\rho_{\mathrm{i}}) + S(\nu_{\mathrm{i}})$$

Calculations (1/2)

First,

$$
\begin{aligned}
S(\nu_i) &:= -\operatorname{Tr}(\nu_i \log \nu_i) = -\operatorname{Tr}\left(\nu_i \log\left(\frac{e^{-\beta H_{\mathcal{R}}}}{Z}\right)\right) \\
&= -\operatorname{Tr}\left(\nu_i \log\left(e^{-\beta H_{\mathcal{R}}}\right)\right) + \log Z \cdot \cancelto{1}{\operatorname{Tr}(\nu_i)} \\
&= \beta \operatorname{Tr}(\nu_i H_{\mathcal{R}}) + \log Z
\end{aligned}
$$

Next,

$$S(\omega_U) = S(\omega_i) = S(\rho_i) + S(\nu_i)$$

Calculations (1/2)

First,

$$
\begin{aligned}
S(\nu_{\mathrm{i}}) :&= -\operatorname{Tr}(\nu_{\mathrm{i}} \log \nu_{\mathrm{i}}) = -\operatorname{Tr}\left(\nu_{\mathrm{i}} \log\left(\frac{e^{-\beta H_{\mathcal{R}}}}{Z}\right)\right) \\
&= -\operatorname{Tr}\left(\nu_{\mathrm{i}} \log\left(e^{-\beta H_{\mathcal{R}}}\right)\right) + \log Z \cdot \cancelto{1}{\operatorname{Tr}(\nu_{\mathrm{i}})} \\
&= \beta \operatorname{Tr}(\nu_{\mathrm{i}} H_{\mathcal{R}}) + \log Z
\end{aligned}
$$

Next,

$$S(\omega_U) = S(\omega_{\mathrm{i}}) = S(\rho_{\mathrm{i}}) + S(\nu_{\mathrm{i}})$$

Calculations (1/2)

First,

$$
\begin{aligned}
S(\nu_\mathrm{i}) :&= -\operatorname{Tr}(\nu_\mathrm{i}\log\nu_\mathrm{i}) = -\operatorname{Tr}\left(\nu_\mathrm{i}\log\left(\frac{e^{-\beta H_\mathcal{R}}}{Z}\right)\right)\\
&= -\operatorname{Tr}\left(\nu_\mathrm{i}\log\left(e^{-\beta H_\mathcal{R}}\right)\right) + \log Z\cdot \cancelto{1}{\operatorname{Tr}(\nu_\mathrm{i})}\\
&= \beta\operatorname{Tr}(\nu_\mathrm{i} H_\mathcal{R}) + \log Z
\end{aligned}
$$

Next,

$$
S(\omega_U) = S(\omega_\mathrm{i}) = S(\rho_\mathrm{i}) + S(\nu_\mathrm{i})
$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$\begin{aligned}
\sigma &= \operatorname{Tr}(\omega_U(\log\omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \operatorname{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \operatorname{Tr}(\omega_U(\log\rho_U + \log\nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \operatorname{Tr}(\rho_U \log\rho_U) - \operatorname{Tr}(\nu_U \log\nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta \operatorname{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta \operatorname{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad\qquad + S(\rho_U) + \beta \operatorname{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta\left(\operatorname{Tr}(\nu_U H_{\mathcal{R}}) - \operatorname{Tr}(\nu_i H_{\mathcal{R}})\right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$\begin{aligned}
\sigma &= \mathrm{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U(\log \rho_U + \log \nu_i)) \\
&= -S(\rho_i) - S(\nu_i) - \mathrm{Tr}(\rho_U \log \rho_U) - \mathrm{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta\,\mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta\,\mathrm{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad + S(\rho_U) + \beta\,\mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta\left(\mathrm{Tr}(\nu_U H_{\mathcal{R}}) - \mathrm{Tr}(\nu_i H_{\mathcal{R}})\right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$\begin{aligned}
\sigma &= \mathrm{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U(\log \rho_U + \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \mathrm{Tr}(\rho_U \log \rho_U) - \mathrm{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta \,\mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta \,\mathrm{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad\qquad + S(\rho_U) + \beta \,\mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta \left(\mathrm{Tr}(\nu_U H_{\mathcal{R}}) - \mathrm{Tr}(\nu_i H_{\mathcal{R}})\right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$\begin{aligned}
\sigma &= \mathrm{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U(\log \rho_U + \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \mathrm{Tr}(\rho_U \log \rho_U) - \mathrm{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta\, \mathrm{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta\left(\mathrm{Tr}(\nu_U H_{\mathcal{R}}) - \mathrm{Tr}(\nu_i H_{\mathcal{R}})\right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$
\begin{aligned}
\sigma &= \mathrm{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U(\log \rho_U + \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \mathrm{Tr}(\rho_U \log \rho_U) - \mathrm{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta\, \mathrm{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta\, (\mathrm{Tr}(\nu_U H_{\mathcal{R}}) - \mathrm{Tr}(\nu_i H_{\mathcal{R}})) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}
$$

Calculations (2/2)

Define $\sigma = S(\omega_U|\rho_U \otimes \nu_i)$. Then

$$\begin{aligned}
\sigma &= \mathrm{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U(\log \rho_U + \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \mathrm{Tr}(\rho_U \log \rho_U) - \mathrm{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta\, \mathrm{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta \left(\mathrm{Tr}(\nu_U H_{\mathcal{R}}) - \mathrm{Tr}(\nu_i H_{\mathcal{R}})\right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$\begin{aligned}
\sigma &= \mathrm{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \mathrm{Tr}(\omega_U(\log \rho_U + \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \mathrm{Tr}(\rho_U \log \rho_U) - \mathrm{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta\, \mathrm{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad\qquad + S(\rho_U) + \beta\, \mathrm{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta\left(\mathrm{Tr}(\nu_U H_{\mathcal{R}}) - \mathrm{Tr}(\nu_i H_{\mathcal{R}})\right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$
\begin{aligned}
\sigma &= \operatorname{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \operatorname{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \operatorname{Tr}(\omega_U(\log \rho_U + \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \operatorname{Tr}(\rho_U \log \rho_U) - \operatorname{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta \operatorname{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta \operatorname{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad\quad + S(\rho_U) + \beta \operatorname{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta \left(\operatorname{Tr}(\nu_U H_{\mathcal{R}}) - \operatorname{Tr}(\nu_i H_{\mathcal{R}})\right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}
$$

Calculations (2/2)

Define $\sigma = S(\omega_U | \rho_U \otimes \nu_i)$. Then

$$\begin{aligned}
\sigma &= \operatorname{Tr}(\omega_U(\log \omega_U - \log(\rho_U \otimes \nu_i))) \\
&= -S(\omega_U) - \operatorname{Tr}(\omega_U \log(\rho_U \otimes \nu_i)) \\
&= -S(\omega_U) - \operatorname{Tr}(\omega_U(\log \rho_U + \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) - \operatorname{Tr}(\rho_U \log \rho_U) - \operatorname{Tr}(\nu_U \log \nu_i) \\
&= -S(\rho_i) - S(\nu_i) + S(\rho_U) + \beta \operatorname{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) - \beta \operatorname{Tr}(\nu_i H_{\mathcal{R}}) - \log Z \\
&\qquad\qquad + S(\rho_U) + \beta \operatorname{Tr}(\nu_U H_{\mathcal{R}}) + \log Z \\
&= -S(\rho_i) + S(\rho_U) + \beta \left(\operatorname{Tr}(\nu_U H_{\mathcal{R}}) - \operatorname{Tr}(\nu_i H_{\mathcal{R}}) \right) \\
&= -\Delta S + \beta \Delta Q
\end{aligned}$$

Landauer's Bound

We are left with

$$\sigma + \Delta S = \beta \Delta Q$$

Using that relative entropies are non-negative,

$$\Delta S \leq \beta \Delta Q$$

Outline

Infinite dimensional discussion (1 of 1)

- Now, the quantum system is described by a C^* dynamical system $(\mathcal{O}, \tau)$, where $\mathcal{O}$ is a C^* algebra with unit $\mathbb{1}$ and τ a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{O}$.
  - A C^*-algebra is a $*$-algebra $\mathcal{A}$ with the property that for all $A \in \mathcal{A}$, $\|A^*A\| = \|A\|^2$, and $\mathcal{A}$ is complete with respect to the norm topology.
- The observables of the system are elements of $\mathcal{O}$ and τ describes their time evolution in the Heisenberg picture.
  - Heisenberg picture: evolve the observables instead of the states
- A state is a positive linear functional ω on $\mathcal{O}$ with $\omega(\mathbb{1}) = 1$; it is τ-invariant if $\omega \circ \tau^t = \omega$ for all $t \in \mathbb{R}$ (e.g. a thermal equilibrium state is τ-invariant).

Infinite dimensional discussion (1 of 1)

- Now, the quantum system is described by a C^* dynamical system $(\mathcal{O}, \tau)$, where $\mathcal{O}$ is a C^* algebra with unit $\mathbb{1}$ and τ a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{O}$.
  - A C^*-algebra is a $*$-algebra $\mathcal{A}$ with the property that for all $A \in \mathcal{A}$, $\|A^*A\| = \|A\|^2$, and $\mathcal{A}$ is complete with respect to the norm topology.
- The observables of the system are elements of $\mathcal{O}$ and τ describes their time evolution in the Heisenberg picture.
  - Heisenberg picture: evolve the observables instead of the states
- A state is a positive linear functional ω on $\mathcal{O}$ with $\omega(\mathbb{1}) = 1$; it is τ-invariant if $\omega \circ \tau^t = \omega$ for all $t \in \mathbb{R}$ (e.g. a thermal equilibrium state is τ-invariant).

Infinite dimensional discussion (1 of 1)

- Now, the quantum system is described by a C^* dynamical system $(\mathcal{O}, \tau)$, where $\mathcal{O}$ is a C^* algebra with unit $\mathbb{1}$ and τ a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{O}$.
  - A C^*-algebra is a $*$-algebra $\mathcal{A}$ with the property that for all $A \in \mathcal{A}$, $\|A^*A\| = \|A\|^2$, and $\mathcal{A}$ is complete with respect to the norm topology.
- The observables of the system are elements of $\mathcal{O}$ and τ describes their time evolution in the Heisenberg picture.
  - Heisenberg picture: evolve the observables instead of the states
- A state is a positive linear functional ω on $\mathcal{O}$ with $\omega(\mathbb{1}) = 1$; it is τ-invariant if $\omega \circ \tau^t = \omega$ for all $t \in \mathbb{R}$ (e.g. a thermal equilibrium state is τ-invariant).

Infinite dimensional discussion (1 of 1)

- Now, the quantum system is described by a C^* dynamical system $(\mathcal{O}, \tau)$, where $\mathcal{O}$ is a C^* algebra with unit $\mathbb{1}$ and τ a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{O}$.
  - A C^*-algebra is a $*$-algebra $\mathcal{A}$ with the property that for all $A \in \mathcal{A}$, $\|A^*A\| = \|A\|^2$, and $\mathcal{A}$ is complete with respect to the norm topology.
- The observables of the system are elements of $\mathcal{O}$ and τ describes their time evolution in the Heisenberg picture.
  - Heisenberg picture: evolve the observables instead of the states
- A state is a positive linear functional ω on $\mathcal{O}$ with $\omega(\mathbb{1}) = 1$; it is τ-invariant if $\omega \circ \tau^t = \omega$ for all $t \in \mathbb{R}$ (e.g. a thermal equilibrium state is τ-invariant).

Infinite dimensional discussion (1 of 1)

- Now, the quantum system is described by a C^* dynamical system $(\mathcal{O}, \tau)$, where $\mathcal{O}$ is a C^* algebra with unit $\mathbb{1}$ and τ a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{O}$.
  - A C^*-algebra is a $*$-algebra $\mathcal{A}$ with the property that for all $A \in \mathcal{A}$, $\|A^*A\| = \|A\|^2$, and $\mathcal{A}$ is complete with respect to the norm topology.
- The observables of the system are elements of $\mathcal{O}$ and τ describes their time evolution in the Heisenberg picture.
  - Heisenberg picture: evolve the observables instead of the states
- A state is a positive linear functional ω on $\mathcal{O}$ with $\omega(\mathbb{1}) = 1$; it is τ-invariant if $\omega \circ \tau^t = \omega$ for all $t \in \mathbb{R}$ (e.g. a thermal equilibrium state is τ-invariant).

Outline

Summary

- In 1961, Landauer posited his principle on physical grounds
- Recently it has been proven very generally using quantum statistical mechanics [JP14]
  - The much easier proof of the finite-dimensional case was presented today

Summary

- In 1961, Landauer posited his principle on physical grounds
- Recently it has been proven very generally using quantum statistical mechanics [JP14]
  - The much easier proof of the finite-dimensional case was presented today

Summary

- In 1961, Landauer posited his principle on physical grounds
- Recently it has been proven very generally using quantum statistical mechanics [JP14]
  - The much easier proof of the finite-dimensional case was presented today

References I

[JP14] Vojkan Jakšić and Claude-Alain Pillet. "A note on the Landauer principle in quantum statistical mechanics". In: *Journal of Mathematical Physics* 55.7, 075210 (2014).

[Lan61] R. Landauer. "Irreversibility and Heat Generation in the Computing Process". In: *IBM Journal of Research and Development* 5.3 (1961), pp. 183–191.

[NC04] Micheal Neilsen and Isaac Chuang. *Quantum Computation and Quantum Information (Cambridge Series on Information and the Natural Sciences)*. Cambridge University Press, Jan. 2004. ISBN: 0521635039.

References II

[RW13] D. Reeb and M. M. Wolf. "(Im-)Proving Landauer's Principle". In: *ArXiv e-prints* (June 2013). arXiv: 1306.4352 [quant-ph].

[Jak+11] V. Jakšić et al. "Entropic Fluctuations in Quantum Statistical Mechanics. An Introduction". In: *ArXiv e-prints* (June 2011). arXiv: 1106.3786 [math-ph].

XY spin chain (1 of 2)

- The reservoir $\mathcal{R}$ is described by a finite dimensional Hilbert space $\mathcal{H}_\mathcal{R} = (\mathbb{C}^2)^{\otimes M}$, equipped with the Hamiltonian

$$H_\mathcal{R} = \frac{1}{2}\sum_{x\in[1,M[} J_x(\sigma_x^{(1)}\sigma_{x+1}^{(1)} + \sigma_x^{(2)}\sigma_{x+1}^{(2)}) + \frac{1}{2}\sum_{x\in[1,M]} \lambda_x\sigma_x^{(3)}$$

  where the $\sigma_x^{(j)}$ are the three Pauli matrices: traceless, self-adjoint 2×2 matrices corresponding to the observables for spin-1/2 particles.
- The system $\mathcal{S}$ has $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$ is just another spin-1/2 particle; the system's Hamiltonian is given by $H_\mathcal{S} = \lambda_0\sigma_0^{(3)}$.

XY spin chain (1 of 2)

- The reservoir $\mathcal{R}$ is described by a finite dimensional Hilbert space $\mathcal{H}_\mathcal{R} = (\mathbb{C}^2)^{\otimes M}$, equipped with the Hamiltonian

$$H_\mathcal{R} = \frac{1}{2} \sum_{x \in [1,M[} J_x(\sigma_x^{(1)}\sigma_{x+1}^{(1)} + \sigma_x^{(2)}\sigma_{x+1}^{(2)}) + \frac{1}{2} \sum_{x \in [1,M]} \lambda_x \sigma_x^{(3)}$$

where the $\sigma_x^{(j)}$ are the three Pauli matrices: traceless, self-adjoint 2×2 matrices corresponding to the observables for spin-$1/2$ particles.

- The system $\mathcal{S}$ has $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$ is just another spin-$1/2$ particle; the system's Hamiltonian is given by $H_\mathcal{S} = \lambda_0 \sigma_0^{(3)}$.

XY spin chain (1 of 2)

- The reservoir $\mathcal{R}$ is described by a finite dimensional Hilbert space $\mathcal{H}_{\mathcal{R}} = (\mathbb{C}^2)^{\otimes M}$, equipped with the Hamiltonian

$$H_{\mathcal{R}} = \frac{1}{2} \sum_{x \in [1,M[} J_x(\sigma_x^{(1)}\sigma_{x+1}^{(1)} + \sigma_x^{(2)}\sigma_{x+1}^{(2)}) + \frac{1}{2} \sum_{x \in [1,M]} \lambda_x \sigma_x^{(3)}$$

  where the $\sigma_x^{(j)}$ are the three Pauli matrices: traceless, self-adjoint 2×2 matrices corresponding to the observables for spin-$1/2$ particles.
- The system $\mathcal{S}$ has $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$ is just another spin-$1/2$ particle; the system's Hamiltonian is given by $H_{\mathcal{S}} = \lambda_0 \sigma_0^{(3)}$.

XY spin chain (1 of 2)

- The reservoir $\mathcal{R}$ is described by a finite dimensional Hilbert space $\mathcal{H}_\mathcal{R} = (\mathbb{C}^2)^{\otimes M}$, equipped with the Hamiltonian

$$H_\mathcal{R} = \frac{1}{2} \sum_{x\in[1,M[} J_x(\sigma_x^{(1)}\sigma_{x+1}^{(1)} + \sigma_x^{(2)}\sigma_{x+1}^{(2)}) + \frac{1}{2} \sum_{x\in[1,M]} \lambda_x \sigma_x^{(3)}$$

  where the $\sigma_x^{(j)}$ are the three Pauli matrices: traceless, self-adjoint 2×2 matrices corresponding to the observables for spin-$1/2$ particles.
- The system $\mathcal{S}$ has $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$ is just another spin-$1/2$ particle; the system's Hamiltonian is given by $H_\mathcal{S} = \lambda_0 \sigma_0^{(3)}$.

XY spin chain (2 of 2)

- We consider the time evolution unitary U generated by the Hamiltonian $H = H_{\mathcal{R}} + K(t) + H_{\mathcal{S}}$ via Schödinger's equation, where $K(t) = J_0(t)\left[\sigma_0^{(1)}\sigma_1^{(1)} + \sigma_0^{(2)}\sigma_1^{(2)}\right]$ is an interaction term.
- My goal is to look in more detail about how Landauer's bound comes about for this system, especially when the limit $M \to \infty$ is considered (recall: M is number of spin-1/2 particles in the chain).

XY spin chain (2 of 2)

- We consider the time evolution unitary U generated by the Hamiltonian $H = H_{\mathcal{R}} + K(t) + H_{\mathcal{S}}$ via Schödinger's equation, where $K(t) = J_0(t)\left[\sigma_0^{(1)}\sigma_1^{(1)} + \sigma_0^{(2)}\sigma_1^{(2)}\right]$ is an interaction term.
- My goal is to look in more detail about how Landauer's bound comes about for this system, especially when the limit $M \to \infty$ is considered (recall: M is number of spin-$1/2$ particles in the chain).