Landauer's Principle in Repeated Interaction Systems

Eric Hanson[1]

Alain Joye[2], Yan Pautrat[3], Renaud Raquépas[1]

[1]McGill University, Montréal, QC,

[2]Institut Fourier, Saint-Martin d'Hères, France

[3]Université Paris-Saclay, Orsay, France

July 13, 2016

Outline

Three ingredients

1. Landauer's Principle
2. Adiabatic theorems
3. Repeated Interaction Systems

Combining the ingredients

Two tools

1. An adiabatic theorem for RIS
2. Perturbation of relative entropy

Entropy production of RIS in adiabatic limit

Outline

Three ingredients

1. Landauer's Principle
2. Adiabatic theorems
3. Repeated Interaction Systems

Combining the ingredients

Two tools

1. An adiabatic theorem for RIS
2. Perturbation of relative entropy

Entropy production of RIS in adiabatic limit

our work

arxiv/1510.00533

Outline

Three ingredients

1. Landauer's Principle
2. Adiabatic theorems
3. Repeated Interaction Systems

Combining the ingredients

Two tools

1. An adiabatic theorem for RIS
2. Perturbation of relative entropy

Entropy production of RIS in adiabatic limit

Outline

Landauer's setup (1/4)

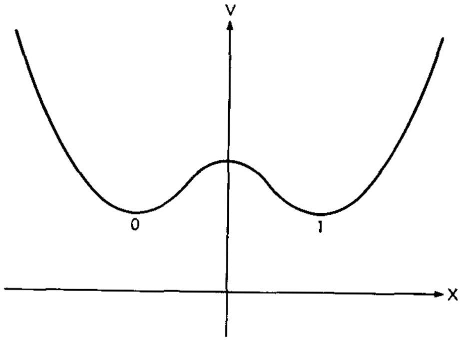


Figure 1 **Bistable potential well.**
x is a generalized coordinate representing quantity which is switched.

Landauer's setup (1/4)

Goal: Erasure process: $E : \{0,1\} \to \{0\}$.

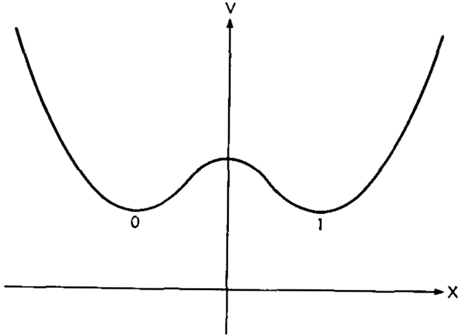
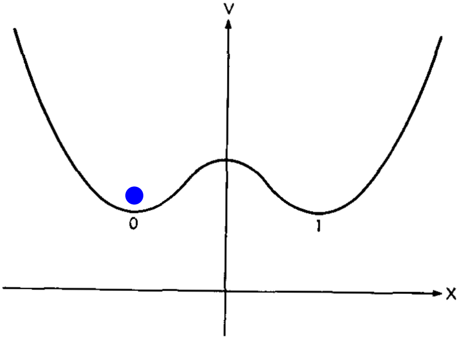


Figure 1 **Bistable potential well.**
x is a generalized coordinate representing quantity which is switched.

Landauer's setup (2/4)

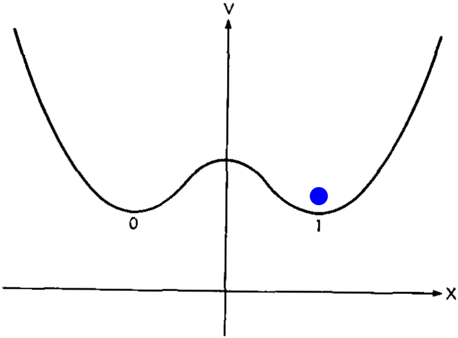
$0 \to 0$



If the system is in 0, we don't have to do anything to erase.

Landauer's setup (3/4)

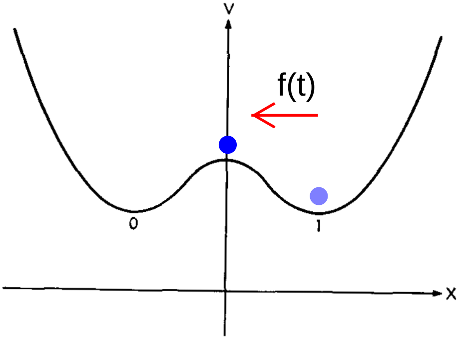
$1 \to 0$



Now consider if the system starts in state 1

Landauer's setup (3/4)

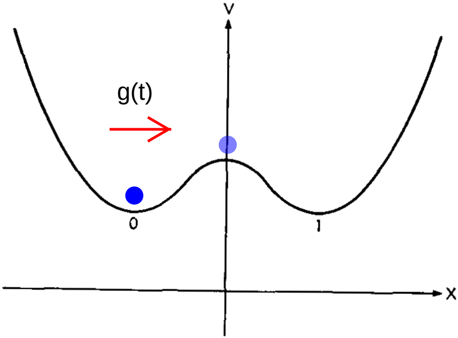
$1 \to 0$



To get to zero, we could apply a force $f(t)$ to the left

Landauer's setup (3/4)

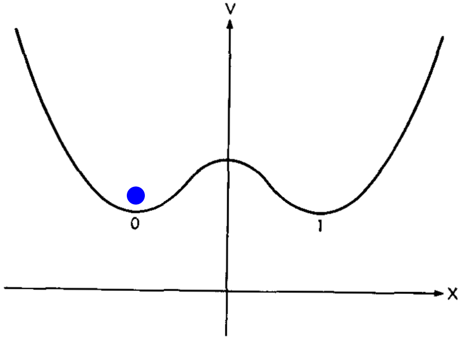
$1 \to 0$



Then remove the energy we added with a force $g(t)$ to the right

Landauer's setup (3/4)

$1 \to 0$



Resulting in the system "erased" in state zero without any net energy cost.

Landauer's setup (4/4)

Landauer's formulation

Wrong.

Landauer's setup (4/4)

Landauer's formulation

Wrong.

- We used two different processes depending on the initial state

Landauer's setup (4/4)

Landauer's formulation

Wrong.

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_{i} to the given final state ρ_{f}

Landauer's setup (4/4)

Landauer's formulation

Wrong.

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_{i} to the given final state ρ_{f}
  - Landauer's motivation [Lan61]: computers operate as a function of circuit connections, not the specific data being handled

Landauer's setup (4/4)

Landauer's formulation

Wrong.

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_{i} to the given final state ρ_{f}
  - Landauer's motivation [Lan61]: computers operate as a function of circuit connections, not the specific data being handled
- Landauer 1961: If we had friction, this would work (with $g(t) \equiv 0$)

Landauer's setup (4/4)

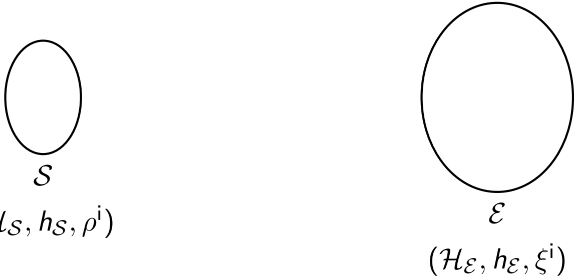
Landauer's formulation

Wrong.

- We used two different processes depending on the initial state
- Our erasure operation should take any initial state ρ_i to the given final state ρ_f
  - Landauer's motivation [Lan61]: computers operate as a function of circuit connections, not the specific data being handled
- Landauer 1961: If we had friction, this would work (with $g(t) \equiv 0$)
  - On the other hand, our erasure map E cuts the size of phase space; since entropy shouldn't decrease, we must have heat output.

Modern (finite-dimensional) quantum formulation (1/3)

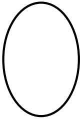
The process



Initial joint state: $\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}$.

Modern (finite-dimensional) quantum formulation (1/3)

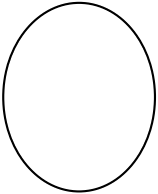
The process



$\mathcal{S}$

$(\mathcal{H}_{\mathcal{S}}, h_{\mathcal{S}}, \rho^{\mathrm{i}})$

Arbitrary ρ^{i}



$\mathcal{E}$

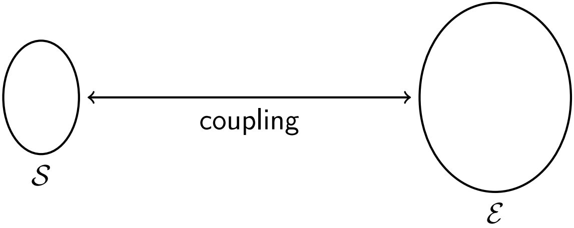
$(\mathcal{H}_{\mathcal{E}}, h_{\mathcal{E}}, \xi^{\mathrm{i}})$

Thermal: $\xi^{\mathrm{i}} = \exp(-\beta h_{\mathcal{E}})/\operatorname{Tr}(\ldots)$

Initial joint state: $\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}$.

Modern (finite-dimensional) quantum formulation (1/3)

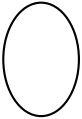
The process



Time evolution by unitary U...

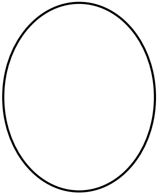
Modern (finite-dimensional) quantum formulation (1/3)

The process



$\mathcal{S}$

$$\rho^{\mathrm{f}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{E}}}(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*)$$



$\mathcal{E}$

$$\xi^{\mathrm{f}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{S}}}(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*)$$

Final joint state: $U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*$.

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr}\rho\log\rho, \qquad S(\eta|\nu) := \operatorname{Tr}\big(\eta(\log\eta - \log\nu)\big)$

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr} \rho \log \rho, \qquad S(\eta|\nu) := \operatorname{Tr}\left(\eta(\log \eta - \log \nu)\right) \geq 0.$

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr}\rho\log\rho, \qquad S(\eta|\nu) := \operatorname{Tr}\big(\eta(\log\eta - \log\nu)\big) \geq 0.$

- Entropy change of system: $\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}})$

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr}\rho\log\rho, \qquad S(\eta|\nu) := \operatorname{Tr}\big(\eta(\log\eta - \log\nu)\big) \geq 0.$

- Entropy change of system: $\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}})$
- Energy change of reservoir: $\Delta Q_{\mathcal{E}} := \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{f}}) - \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{i}})$.

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr}\rho\log\rho, \qquad S(\eta|\nu) := \operatorname{Tr}\big(\eta(\log\eta - \log\nu)\big) \geq 0.$

- Entropy change of system: $\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}})$
- Energy change of reservoir: $\Delta Q_{\mathcal{E}} := \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{f}}) - \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{i}})$.
- Entropy production: $\sigma := S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}\, U^* | \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}) \geq 0$

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr}\rho\log\rho$, $\qquad S(\eta|\nu) := \operatorname{Tr}\big(\eta(\log\eta - \log\nu)\big) \geq 0.$

- Entropy change of system: $\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}})$
- Energy change of reservoir: $\Delta Q_{\mathcal{E}} := \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{f}}) - \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{i}})$.
- Entropy production: $\boxed{\sigma} := S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}\, U^*|\rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}) \geq 0$

key object #1

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr} \rho \log \rho, \qquad S(\eta|\nu) := \operatorname{Tr}\left(\eta(\log \eta - \log \nu)\right) \geq 0.$

- Entropy change of system: $\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}})$
- Energy change of reservoir: $\Delta Q_{\mathcal{E}} := \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{f}}) - \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{i}})$.
- Entropy production: $\boxed{\sigma} := S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}\, U^* | \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}) \geq 0$

  key object #1

- Computation of σ using ξ^{i} is Gibbs:

$$\begin{aligned}
\sigma &= -S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*) - \operatorname{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \left(\log \rho^{\mathrm{f}} \otimes \mathrm{Id}\right)\right) \\
&\qquad - \operatorname{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \left(\mathrm{Id} \otimes \log \xi^{\mathrm{i}}\right)\right) \\
&= -S(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}) + S(\rho^{\mathrm{f}}) - \operatorname{Tr}(\xi^{\mathrm{f}} \log \xi^{\mathrm{i}}) \\
&= -S(\rho^{\mathrm{i}}) - S(\xi^{\mathrm{i}}) + S(\rho^{\mathrm{f}}) - \operatorname{Tr}(\xi^{\mathrm{f}} \log \xi^{\mathrm{i}}) \\
&= -\Delta S_{\mathcal{S}} + \beta \Delta Q_{\mathcal{E}}.
\end{aligned}$$

Modern (finite-dimensional) quantum formulation (2/3)

Quantities of interest

Def: $S(\rho) := -\operatorname{Tr}\rho\log\rho, \qquad S(\eta|\nu) := \operatorname{Tr}\left(\eta(\log\eta - \log\nu)\right) \geq 0.$

- Entropy change of system: $\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}})$
- Energy change of reservoir: $\Delta Q_{\mathcal{E}} := \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{f}}) - \operatorname{Tr}(h_{\mathcal{E}}\xi^{\mathrm{i}})$.
- Entropy production: $\boxed{\sigma} := S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}\, U^* | \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}) \geq 0$

  key object #1
- Computation of σ using ξ^{i} is Gibbs:

$$\begin{aligned}\sigma = -S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*) &- \operatorname{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \left(\log\rho^{\mathrm{f}} \otimes \mathrm{Id}\right)\right) \\ &- \operatorname{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \left(\mathrm{Id} \otimes \log\xi^{\mathrm{i}}\right)\right) \\ = -S(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}) &+ S(\rho^{\mathrm{f}}) - \operatorname{Tr}(\xi^{\mathrm{f}}\log\xi^{\mathrm{i}}) \\ = -S(\rho^{\mathrm{i}}) &- S(\xi^{\mathrm{i}}) + S(\rho^{\mathrm{f}}) - \operatorname{Tr}(\xi^{\mathrm{f}}\log\xi^{\mathrm{i}})\end{aligned}$$

$$\boxed{\sigma = -\Delta S_{\mathcal{S}} + \beta\Delta Q_{\mathcal{E}}.} \qquad \textit{the balance equation}$$

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0$$

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow$$

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.$$

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.}$$ *Landauer's Principle*

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.} \qquad \textit{Landauer's Principle}$$

Special case: Erasure process of qubit: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\rho^{\mathrm{i}} = \frac{1}{2}\mathrm{Id}$, $\rho^{\mathrm{f}} = |0\rangle\langle 0|$. Then $\Delta S_{\mathcal{S}} = \log 2$, and LP becomes

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.} \qquad \textit{Landauer's Principle}$$

Special case: Erasure process of qubit: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\rho^{\mathrm{i}} = \frac{1}{2}\mathrm{Id}$, $\rho^{\mathrm{f}} = |0\rangle\langle 0|$. Then $\Delta S_{\mathcal{S}} = \log 2$, and LP becomes

$$\Delta Q_{\mathcal{E}} \geq T \log 2. \qquad (T = \beta^{-1},\ k_{\mathrm{B}} \equiv 1)$$

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.} \qquad \textit{Landauer's Principle}$$

Special case: Erasure process of qubit: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\rho^{\mathrm{i}} = \frac{1}{2}\mathrm{Id}$, $\rho^{\mathrm{f}} = |0\rangle\langle 0|$. Then $\Delta S_{\mathcal{S}} = \log 2$, and LP becomes

$$\Delta Q_{\mathcal{E}} \geq T \log 2. \qquad (T = \beta^{-1},\ k_{\mathrm{B}} \equiv 1)$$

When do we have the equality $\Delta Q_{\mathcal{E}} = \beta^{-1} \Delta S_{\mathcal{S}}$?

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.} \qquad \textit{Landauer's Principle}$$

Special case: Erasure process of qubit: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\rho^{\mathrm{i}} = \frac{1}{2}\mathrm{Id}$, $\rho^{\mathrm{f}} = |0\rangle\langle 0|$. Then $\Delta S_{\mathcal{S}} = \log 2$, and LP becomes

$$\Delta Q_{\mathcal{E}} \geq T \log 2. \qquad (T = \beta^{-1},\ k_{\mathrm{B}} \equiv 1)$$

When do we have the equality $\Delta Q_{\mathcal{E}} = \beta^{-1} \Delta S_{\mathcal{S}}$?

When $\sigma = 0$, so $U \rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* = \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}$.

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.} \qquad \textit{Landauer's Principle}$$

Special case: Erasure process of qubit: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\rho^{\mathrm{i}} = \frac{1}{2}\mathrm{Id}$, $\rho^{\mathrm{f}} = |0\rangle\langle 0|$. Then $\Delta S_{\mathcal{S}} = \log 2$, and LP becomes

$$\Delta Q_{\mathcal{E}} \geq T \log 2. \qquad (T = \beta^{-1},\ k_{\mathrm{B}} \equiv 1)$$

When do we have the equality $\Delta Q_{\mathcal{E}} = \beta^{-1} \Delta S_{\mathcal{S}}$?

When $\sigma = 0$, so $U \rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* = \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}$. Taking the partial trace, $\xi^{\mathrm{f}} = \xi^{\mathrm{i}}$ so $\Delta Q_{\mathcal{E}} = 0$

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.} \qquad \textit{Landauer's Principle}$$

Special case: Erasure process of qubit: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\rho^{\mathrm{i}} = \frac{1}{2}\mathrm{Id}$, $\rho^{\mathrm{f}} = |0\rangle\langle 0|$. Then $\Delta S_{\mathcal{S}} = \log 2$, and LP becomes

$$\Delta Q_{\mathcal{E}} \geq T \log 2. \qquad (T = \beta^{-1},\ k_{\mathrm{B}} \equiv 1)$$

When do we have the equality $\Delta Q_{\mathcal{E}} = \beta^{-1} \Delta S_{\mathcal{S}}$?

When $\sigma = 0$, so $U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* = \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}$. Taking the partial trace, $\xi^{\mathrm{f}} = \xi^{\mathrm{i}}$ so $\Delta Q_{\mathcal{E}} = 0$, implying $\Delta S_{\mathcal{S}} = 0$.

Modern (finite-dimensional) quantum formulation (3/3)

The Principle

$$\sigma \geq 0 \quad \Longrightarrow \quad \boxed{\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}.} \qquad \textit{Landauer's Principle}$$

Special case: Erasure process of qubit: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\rho^{\mathrm{i}} = \frac{1}{2}\mathrm{Id}$, $\rho^{\mathrm{f}} = |0\rangle\langle 0|$. Then $\Delta S_{\mathcal{S}} = \log 2$, and LP becomes

$$\Delta Q_{\mathcal{E}} \geq T \log 2. \qquad (T = \beta^{-1},\ k_{\mathrm{B}} \equiv 1)$$

When do we have the equality $\Delta Q_{\mathcal{E}} = \beta^{-1} \Delta S_{\mathcal{S}}$?

When $\sigma = 0$, so $U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* = \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}$. Taking the partial trace, $\xi^{\mathrm{f}} = \xi^{\mathrm{i}}$ so $\Delta Q_{\mathcal{E}} = 0$, implying $\Delta S_{\mathcal{S}} = 0$.

In fact, tighter bounds exist in finite dimensions [RW14].

Outline

The adiabatic limit

- In this unitary time evolution set up, we write Schrödinger's equation

$$i\frac{\mathrm{d}}{\mathrm{d}s}U(s) = h(s)U(s),\ s \in [0,1],\ \text{with}\ U(0) = \mathrm{Id}.$$

The adiabatic limit

- In this unitary time evolution set up, we write Schrödinger's equation

$$i\frac{\mathrm{d}}{\mathrm{d}s}U(s) = h(s)U(s),\ s \in [0,1],\ \text{with } U(0) = \mathrm{Id}.$$

- Then *adiabatic limit* concerns the solution $U_T(s)$ of the rescaled Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}U_T(t) = h(t/T)\,U_T(t),\ t \in [0,T],\ \text{with } U_T(0) = \mathrm{Id},$$

in the limit $T \to \infty$.

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50].

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50]. Assumptions:

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50]. Assumptions:

1. $e(s)$ be a continuous parametrization of an eigenvalue of $h(s)$ for $s \in [0, 1]$.

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50]. Assumptions:

1. $e(s)$ be a continuous parametrization of an eigenvalue of $h(s)$ for $s \in [0, 1]$.
2. $e(s)$ separated from the rest of the spectrum by a constant gap.

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50]. Assumptions:

1. $e(s)$ be a continuous parametrization of an eigenvalue of $h(s)$ for $s \in [0, 1]$.
2. $e(s)$ separated from the rest of the spectrum by a constant gap.
3. Let $P(s)$ be the projection onto the eigenspace of $e(s)$. Then we assume $s \mapsto P(s)$ is C^2.

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50]. Assumptions:

1. $e(s)$ be a continuous parametrization of an eigenvalue of $h(s)$ for $s \in [0, 1]$.
2. $e(s)$ separated from the rest of the spectrum by a constant gap.
3. Let $P(s)$ be the projection onto the eigenspace of $e(s)$. Then we assume $s \mapsto P(s)$ is C^2.

Kato: Then there exists a unitary operator-valued function $[0, T] \ni t \mapsto W(t)$ that

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50]. Assumptions:

1. $e(s)$ be a continuous parametrization of an eigenvalue of $h(s)$ for $s \in [0, 1]$.
2. $e(s)$ separated from the rest of the spectrum by a constant gap.
3. Let $P(s)$ be the projection onto the eigenspace of $e(s)$. Then we assume $s \mapsto P(s)$ is C^2.

Kato: Then there exists a unitary operator-valued function $[0, T] \ni t \mapsto W(t)$ that

1. Approximate time evolution on Ran $P(0)$:

$$\Big(U_T(t) - \exp\big(-\mathrm{i}T\int_0^{t/T} e(s)\,\mathrm{d}s\big)\,W(t)\Big)\,P(0) = O(T^{-1})$$

uniformly in $t \in [0, T]$.

Kato's adiabatic theorem

Many, but Kato's is representative [Kat50]. Assumptions:

1. $e(s)$ be a continuous parametrization of an eigenvalue of $h(s)$ for $s \in [0, 1]$.
2. $e(s)$ separated from the rest of the spectrum by a constant gap.
3. Let $P(s)$ be the projection onto the eigenspace of $e(s)$. Then we assume $s \mapsto P(s)$ is C^2.

Kato: Then there exists a unitary operator-valued function $[0, T] \ni t \mapsto W(t)$ that

1. Approximate time evolution on Ran $P(0)$:
$$\Big(U_T(t) - \exp\big(-\mathrm{i}T\int_0^{t/T} e(s)\,\mathrm{d}s\big)\,W(t)\Big)\,P(0) = O(T^{-1})$$
uniformly in $t \in [0, T]$.
2. Intertwine with P: $W(t)P(0) = P(t/T)W(t)$.

Adiabatic limit of Landauer's Principle

At fixed $T > 0$, we can consider the quantities ΔS_T, ΔQ_T defined through the time evolution $U_T = U_T(1)$. Then we obtain our balance equation

$$\Delta S_T + \sigma_T = \beta \Delta Q_T,$$

Adiabatic limit of Landauer's Principle

At fixed $T > 0$, we can consider the quantities ΔS_T, ΔQ_T defined through the time evolution $U_T = U_T(1)$. Then we obtain our balance equation

$$\Delta S_T + \sigma_T = \beta \Delta Q_T,$$

with

$$\sigma_T = S(U_T(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}})U_T^* | \rho_T^{\mathrm{f}} \otimes \xi_T^{\mathrm{i}}).$$

Adiabatic limit of Landauer's Principle

At fixed $T > 0$, we can consider the quantities ΔS_T, ΔQ_T defined through the time evolution $U_T = U_T(1)$. Then we obtain our balance equation

$$\Delta S_T + \sigma_T = \beta \Delta Q_T,$$

with

$$\sigma_T = S(U_T(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}})U_T^* | \rho_T^{\mathrm{f}} \otimes \xi_T^{\mathrm{i}}).$$

- In [JP14], the authors use the Avron-Elgart adiabatic theorem and show $\sigma_T \to 0$, when the reservoir is infinite dimensional, with the help of an ergodicity assumption.

Outline

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}.

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.
2. Then $\mathcal{S}$ interacts with $\mathcal{E}_k$:

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.
2. Then $\mathcal{S}$ interacts with $\mathcal{E}_k$:
   - which has initial state ξ_k^{i}

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.
2. Then $\mathcal{S}$ interacts with $\mathcal{E}_k$:
   - which has initial state ξ_k^{i}
   - for a time τ_k

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.
2. Then $\mathcal{S}$ interacts with $\mathcal{E}_k$:
   - which has initial state ξ_k^{i}
   - for a time τ_k
   - via a coupling $\lambda\, v_k \in \mathcal{B}_{\text{s.a.}}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$ (with $\lambda \in \mathbb{R}_+$).

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.
2. Then $\mathcal{S}$ interacts with $\mathcal{E}_k$:
   - which has initial state ξ_k^{i}
   - for a time τ_k
   - via a coupling $\lambda\, v_k \in \mathcal{B}_{\mathrm{s.a.}}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$ (with $\lambda \in \mathbb{R}_+$).

$$\rightsquigarrow U_k := \exp\big(-i\tau_k(h_{\mathcal{S}} \otimes \mathrm{Id} + \mathrm{Id} \otimes h_{\mathcal{E}_k} + \lambda\, v_k)\big)$$

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.
2. Then $\mathcal{S}$ interacts with $\mathcal{E}_k$:
   - which has initial state ξ_k^{i}
   - for a time τ_k
   - via a coupling $\lambda\, v_k \in \mathcal{B}_{\text{s.a.}}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$ (with $\lambda \in \mathbb{R}_+$).

   $$\rightsquigarrow U_k := \exp\big(-i\tau_k(h_{\mathcal{S}} \otimes \mathrm{Id} + \mathrm{Id} \otimes h_{\mathcal{E}_k} + \lambda\, v_k)\big)$$
3. Joint final state $U_k \rho_{k-1} \otimes \xi_k^{\mathsf{i}} U_k^*$.

RIS Setup (words)

System $\mathcal{S}$ starts in state ρ^{i}. $\mathcal{S}$ interacts with a chain of probes $\{\mathcal{E}_k\}_{k=1}^{\infty}$, one at a time as follows.

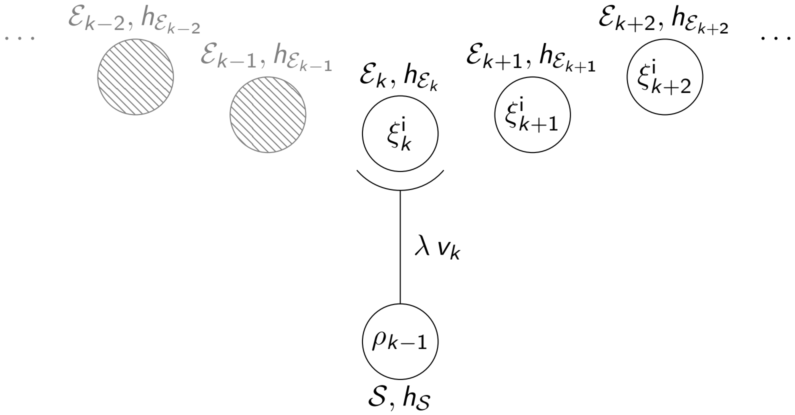
1. Assume after interacting with $k-1$ probes, the system is in state ρ_{k-1}.
2. Then $\mathcal{S}$ interacts with $\mathcal{E}_k$:
   - which has initial state ξ_k^{i}
   - for a time τ_k
   - via a coupling $\lambda\, v_k \in \mathcal{B}_{\text{s.a.}}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$ (with $\lambda \in \mathbb{R}_+$).

   $$\rightsquigarrow U_k := \exp\big(-i\tau_k(h_{\mathcal{S}} \otimes \mathrm{Id} + \mathrm{Id} \otimes h_{\mathcal{E}_k} + \lambda\, v_k)\big)$$

3. Joint final state $U_k \rho_{k-1} \otimes \xi_k^{\mathrm{i}} U_k^*$.
4. Trace out $\mathcal{E}_k$ to obtain the system state

$$\rho_k := \mathrm{Tr}_{\mathcal{E}}\big(U_k(\rho_{k-1} \otimes \xi_k^{\mathrm{i}})U_k^*\big).$$

RIS Setup (schematic)



Simplifications

- The Hilbert space for the system and each probe is finite dimensional.

Simplifications

- The Hilbert space for the system and each probe is finite dimensional.
- Interaction time is constant: $\tau_k \equiv \tau > 0$.

Simplifications

- The Hilbert space for the system and each probe is finite dimensional.
- Interaction time is constant: $\tau_k \equiv \tau > 0$.
- Each probe's initial state is a Gibbs state at some inverse temperature β_k:

$$\xi_k^{\mathrm{i}} = \frac{e^{-\beta_k h_{\mathcal{E}_k}}}{\mathrm{Tr}(e^{-\beta_k h_{\mathcal{E}_k}})}.$$

Favorite example(s) (1/2)

- Qubits: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_{\mathcal{E}} = \mathbb{C}^2$.

Favorite example(s) (1/2)

- Qubits: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_{\mathcal{E}} = \mathbb{C}^2$.
- We define $h_{\mathcal{S}} = E\, a^* a$ and $h_{\mathcal{E}_k} \equiv h_{\mathcal{E}} = E_0\, b^* b$, where a/a^*, resp. b/b^*, are the annihilation/creation operators for $\mathcal{S}$, resp. $\mathcal{E}$:

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* a = b^* b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Favorite example(s) (1/2)

- Qubits: $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$, $\mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_\mathcal{E} = \mathbb{C}^2$.
- We define $h_\mathcal{S} = E\, a^* a$ and $h_{\mathcal{E}_k} \equiv h_\mathcal{E} = E_0\, b^* b$, where a/a^*, resp. b/b^*, are the annihilation/creation operators for $\mathcal{S}$, resp. $\mathcal{E}$:

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* a = b^* b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\rightsquigarrow h_\mathcal{S} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \text{ and } h_{\mathcal{E}_k} \equiv h_\mathcal{E} = \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix}.$$

Favorite example(s) (1/2)

- Qubits: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $\mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_{\mathcal{E}} = \mathbb{C}^2$.
- We define $h_{\mathcal{S}} = E\, a^* a$ and $h_{\mathcal{E}_k} \equiv h_{\mathcal{E}} = E_0\, b^* b$, where a/a^*, resp. b/b^*, are the annihilation/creation operators for $\mathcal{S}$, resp. $\mathcal{E}$:

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* a = b^* b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\rightsquigarrow h_{\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \text{ and } h_{\mathcal{E}_k} \equiv h_{\mathcal{E}} = \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix}.$$

- Take constant potential $v_k \equiv v$.

Favorite example(s) (1/2)

- Qubits: $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$, $\mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_\mathcal{E} = \mathbb{C}^2$.
- We define $h_\mathcal{S} = E\, a^* a$ and $h_{\mathcal{E}_k} \equiv h_\mathcal{E} = E_0\, b^* b$, where a/a^*, resp. b/b^*, are the annihilation/creation operators for $\mathcal{S}$, resp. $\mathcal{E}$:

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* a = b^* b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\rightsquigarrow h_\mathcal{S} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \text{ and } h_{\mathcal{E}_k} \equiv h_\mathcal{E} = \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix}.$$

- Take constant potential $v_k \equiv v$.
- As stated, $\xi_k^\mathrm{i} = \exp(-\beta_k h_{\mathcal{E}_k}) / \operatorname{Tr}(\ldots)$ Gibbs state at inverse temperature β_k.

Favorite example(s) (1/2)

- Qubits: $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$, $\mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_\mathcal{E} = \mathbb{C}^2$.
- We define $h_\mathcal{S} = E\, a^* a$ and $h_{\mathcal{E}_k} \equiv h_\mathcal{E} = E_0\, b^* b$, where a/a^*, resp. b/b^*, are the annihilation/creation operators for $\mathcal{S}$, resp. $\mathcal{E}$:

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* a = b^* b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\rightsquigarrow h_\mathcal{S} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \text{ and } h_{\mathcal{E}_k} \equiv h_\mathcal{E} = \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix}.$$

- Take constant potential $v_k \equiv v$.
- As stated, $\xi_k^{\mathrm{i}} = \exp(-\beta_k h_{\mathcal{E}_k}) / \operatorname{Tr}(\ldots)$ Gibbs state at inverse temperature β_k. $\leftarrow$ only parameter which changes between probes

Favorite example(s) (2/2)

- Two choices of potential for our qubits:

Favorite example(s) (2/2)

- Two choices of potential for our qubits:
  1. Full dipole:

$$v_{\mathrm{FD}} = \frac{1}{2}(a^* + a) \otimes (b^* + b) = \frac{1}{2}(a \otimes b + a^* \otimes b + a \otimes b^* + a^* \otimes b^*).$$

Favorite example(s) (2/2)

- Two choices of potential for our qubits:
  1. Full dipole:

  $$v_{\mathrm{FD}} = \frac{1}{2}(a^* + a) \otimes (b^* + b) = \frac{1}{2}(a \otimes b + a^* \otimes b + a \otimes b^* + a^* \otimes b^*).$$

  2. Rotating wave (common approx. when $|E - E_0| \ll \min\{E, E_0\}$ and $\lambda \ll |E_0|$):

  $$v_{\mathrm{RW}} = \frac{1}{2}(a^* \otimes b + a \otimes b^*).$$

Favorite example(s) (2/2)

- Two choices of potential for our qubits:
  1. Full dipole:
  $$v_{\mathrm{FD}} = \frac{1}{2}(a^* + a) \otimes (b^* + b) = \frac{1}{2}(a \otimes b + a^* \otimes b + a \otimes b^* + a^* \otimes b^*).$$
  2. Rotating wave (common approx. when $|E - E_0| \ll \min\{E, E_0\}$ and $\lambda \ll |E_0|$):
  $$v_{\mathrm{RW}} = \frac{1}{2}(a^* \otimes b + a \otimes b^*).$$

We will consider both potentials.

Reduced dynamics

- We can consider only the dynamics on the system. Define

$$\begin{array}{cccc} \mathcal{L}_k : & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) & \rightarrow & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) \\ & \eta & \mapsto & \mathrm{Tr}_\mathcal{E}\left(U_k(\eta \otimes \xi_k^{\mathrm{i}})U_k^*\right) \end{array}$$

where $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$ are the trace-class operators on $\mathcal{H}_\mathcal{S}$, i.e., all linear operators.

Reduced dynamics

- We can consider only the dynamics on the system. Define

$$\begin{array}{cccc} \underset{\text{key object \#2}}{\boxed{\mathcal{L}_k}}: & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) & \to & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) \\ & \eta & \mapsto & \mathrm{Tr}_\mathcal{E}\left(U_k(\eta \otimes \xi_k^{\mathrm{i}})U_k^*\right) \end{array}$$

where $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$ are the trace-class operators on $\mathcal{H}_\mathcal{S}$, i.e., all linear operators.

Reduced dynamics

- We can consider only the dynamics on the system. Define

$$\begin{array}{cccc} \boxed{\mathcal{L}_k}: & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) & \to & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) \\ \text{key object \#2} & \eta & \mapsto & \mathrm{Tr}_\mathcal{E}\left(U_k(\eta \otimes \xi_k^\mathrm{i})U_k^*\right) \end{array}$$

  where $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$ are the trace-class operators on $\mathcal{H}_\mathcal{S}$, i.e., all linear operators.

- Then

$$\rho_k = \mathcal{L}_k\mathcal{L}_{k-1}\cdots\mathcal{L}_1\rho^\mathrm{i}.$$

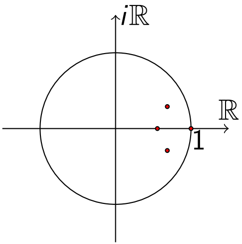
  This is a *Markovian* form for the sequence of states $(\rho_k)_k$.

Reduced dynamics in our favorite examples

- With both v_{RW} and v_{FD}, we've computed a 4x4 matrix representation of the reduced dynamics $\mathcal{L}(\beta)$ in terms of E, E_0, β, τ and λ. Recall only β changes with the step, so we may obtain $\mathcal{L}_k = \mathcal{L}(\beta_k)$.

Reduced dynamics in our favorite examples

- With both v_{RW} and v_{FD}, we've computed a 4x4 matrix representation of the reduced dynamics $\mathcal{L}(\beta)$ in terms of E, E_0, β, τ and λ. Recall only β changes with the step, so we may obtain $\mathcal{L}_k = \mathcal{L}(\beta_k)$.



Numerically obtained eigenvalues of $\mathcal{L}_{\mathrm{FD}}$ with $\lambda = 2$, $\tau = 0.5$, $E_0 = 0.8$, and $E = 0.9$. The evals of $\mathcal{L}_{\mathrm{FD}}$ are independent of β.

Properties of the reduced dynamics

- $\mathcal{L}_k$ is trace preserving

Properties of the reduced dynamics

- $\mathcal{L}_k$ is trace preserving
- $\mathcal{L}_k$ is completely positive: if $\eta \geq 0$ is a matrix on $\mathcal{H}_\mathcal{S} \otimes \mathbb{C}^n$, then $(\mathcal{L}_k \otimes \mathrm{Id})(\eta) \geq 0$, for every $n \in \mathbb{N}$.

Properties of the reduced dynamics

- $\mathcal{L}_k$ is trace preserving
- $\mathcal{L}_k$ is completely positive: if $\eta \geq 0$ is a matrix on $\mathcal{H}_\mathcal{S} \otimes \mathbb{C}^n$, then $(\mathcal{L}_k \otimes \mathrm{Id})(\eta) \geq 0$, for every $n \in \mathbb{N}$.
- Fact: if we choose the trace norm on our domain $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$, (namely, $\|\eta\|_1 = \mathrm{Tr}(\sqrt{\eta^* \eta})$) then the operator norm of $\mathcal{L}_k$ is

$$\|\mathcal{L}_k\| := \sup_{\|\eta\|_1 = 1} \|\mathcal{L}_k \eta\|_1 = 1.$$

Properties of the reduced dynamics

- $\mathcal{L}_k$ is trace preserving
- $\mathcal{L}_k$ is completely positive: if $\eta \geq 0$ is a matrix on $\mathcal{H}_\mathcal{S} \otimes \mathbb{C}^n$, then $(\mathcal{L}_k \otimes \mathrm{Id})(\eta) \geq 0$, for every $n \in \mathbb{N}$.
- Fact: if we choose the trace norm on our domain $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$, (namely, $\|\eta\|_1 = \mathrm{Tr}(\sqrt{\eta^*\eta})$) then the operator norm of $\mathcal{L}_k$ is

$$\|\mathcal{L}_k\| := \sup_{\|\eta\|_1=1} \|\mathcal{L}_k\eta\|_1 = 1.$$

- However, in general, we could have $\|\mathcal{L}_k\| > 1$ when considered as an operator on $(\mathcal{I}_1(\mathcal{H}_\mathcal{S}), \|\cdot\|_2)$, where $\|A\|_2 = \sqrt{\mathrm{Tr}(A^*A)}$.

Outline

Three ingredients

1. Landauer's Principle
2. Adiabatic theorems
3. Repeated Interaction Systems

Combining the ingredients

Two tools

1. An adiabatic theorem for RIS
2. Perturbation of relative entropy

Entropy production of RIS in adiabatic limit

Landauer's Principle at step k of an RIS

- The entropy change of $\mathcal{S}$ and energy change of $\mathcal{E}_k$ at step k is given by

$$\Delta S_k := S(\rho_{k-1}) - S(\rho_k) = S(\rho_{k-1}) - S(\mathcal{L}_k(\rho_{k-1})),$$
$$\Delta Q_k := \mathrm{Tr}\left(h_{\mathcal{E}_k} \underbrace{\mathrm{Tr}_{\mathcal{S}}\big(U_k(\rho_{k-1} \otimes \xi_k^{\mathrm{i}})U_k^*\big)}_{\xi_k^{\mathrm{f}}}\right) - \mathrm{Tr}(h_{\mathcal{E}_k}\xi_k^{\mathrm{i}}),$$

Landauer's Principle at step k of an RIS

- The entropy change of $\mathcal{S}$ and energy change of $\mathcal{E}_k$ at step k is given by

$$\Delta S_k := S(\rho_{k-1}) - S(\rho_k) = S(\rho_{k-1}) - S(\mathcal{L}_k(\rho_{k-1})),$$
$$\Delta Q_k := \mathrm{Tr}\left(h_{\mathcal{E}_k} \underbrace{\mathrm{Tr}_{\mathcal{S}}\big(U_k(\rho_{k-1} \otimes \xi_k^{\mathrm{i}})U_k^*\big)}_{\xi_k^{\mathrm{f}}}\right) - \mathrm{Tr}(h_{\mathcal{E}_k}\xi_k^{\mathrm{i}}),$$

- So the balance equation holds at step k

$$\Delta S_k + \sigma_k = \beta_k \Delta Q_k,$$

with

$$\sigma_k := S\big(U_k(\rho_{k-1} \otimes \xi_k^{\mathrm{i}})U_k^* | \mathcal{L}_k(\rho_{k-1}) \otimes \xi_k^{\mathrm{i}}\big).$$

Adiabatic RIS

- To consider an "adiabatic limit" of an RIS process, we will consider $T \gg 0$ interactions (i.e., $\mathcal{S}$ interacts with $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_T$), and

Adiabatic RIS

- To consider an "adiabatic limit" of an RIS process, we will consider $T \gg 0$ interactions (i.e., $\mathcal{S}$ interacts with $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_T$), and
- We will sample the probe and interaction parameters from C^2 functions. That is, define C^2 functions

$$s \mapsto h_{\mathcal{E}}(s), \qquad s \mapsto \beta(s), \qquad s \mapsto v(s)$$

for $s \in [0, 1]$,

Adiabatic RIS

- To consider an "adiabatic limit" of an RIS process, we will consider $T \gg 0$ interactions (i.e., $\mathcal{S}$ interacts with $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_T$), and
- We will sample the probe and interaction parameters from C^2 functions. That is, define C^2 functions

$$s \mapsto h_{\mathcal{E}}(s), \qquad s \mapsto \beta(s), \qquad s \mapsto v(s)$$

for $s \in [0, 1]$, and choose

$$h_{\mathcal{E}_{k,T}} = h_{\mathcal{E}}(k/T), \qquad \beta_{k,T} = \beta(k/T), \qquad v_{k,T} = v(k/T).$$

LP in RIS in the adiabatic limit (1/2)

- At each step, we have $\sigma_{k,T}$ a relative entropy, which depends on T as it depends on our parameters e.g. $\beta_{k,T} = \beta(k/T)$.

LP in RIS in the adiabatic limit (1/2)

- At each step, we have $\sigma_{k,T}$ a relative entropy, which depends on T as it depends on our parameters e.g. $\beta_{k,T} = \beta(k/T)$.
- We may sum over the T steps and define $\sigma_T^{\text{tot}} := \sum_{k=1}^{T} \sigma_{k,T}$.
  - Depends on T in two ways: each $\sigma_{k,T}$ has parameters depending on T, and we consider T probes.

LP in RIS in the adiabatic limit (1/2)

- At each step, we have $\sigma_{k,T}$ a relative entropy, which depends on T as it depends on our parameters e.g. $\beta_{k,T} = \beta(k/T)$.
- We may sum over the T steps and define $\sigma_T^{\text{tot}} := \sum_{k=1}^{T} \sigma_{k,T}$.
  - Depends on T in two ways: each $\sigma_{k,T}$ has parameters depending on T, and we consider T probes.
- We are interested in $\lim_{T\to\infty} \sigma_T^{\text{tot}}$. Does this vanish?

LP in RIS in the adiabatic limit (2/2)

- We are in a fundamentally different setup than the single-reservoir unitary one considered earlier.

LP in RIS in the adiabatic limit (2/2)

- We are in a fundamentally different setup than the single-reservoir unitary one considered earlier.
- We are also secretly in an infinite dimensional system: each probe has a different Hilbert space, and we are taking the number of probes T to infinity.

LP in RIS in the adiabatic limit (2/2)

- We are in a fundamentally different setup than the single-reservoir unitary one considered earlier.
- We are also secretly in an infinite dimensional system: each probe has a different Hilbert space, and we are taking the number of probes T to infinity.
- Each $\sigma_{k,T} \geq 0$, so we need vanishing entropy production at each step.

LP in RIS in the adiabatic limit (2/2)

- We are in a fundamentally different setup than the single-reservoir unitary one considered earlier.
- We are also secretly in an infinite dimensional system: each probe has a different Hilbert space, and we are taking the number of probes T to infinity.
- Each $\sigma_{k,T} \geq 0$, so we need vanishing entropy production at each step.
  - In fact, want $\sigma_{k,T} = O(1/T^2)$, so $\sigma_T^{\text{tot}} = O(1/T)$.

Strategy

- Our key object:

$$\sigma_{k,T} = S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big),$$

Strategy

- Our key object:

$$\sigma_{k,T} = S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^{*}_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big),$$

where $\rho_{k-1,T} = \mathcal{L}_{k-1,T}\mathcal{L}_{k-2,T} \cdots \mathcal{L}_{1,T}\rho^{\mathrm{i}}$.

Strategy

- Our key object:

$$\sigma_{k,T} = S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big),$$
$$\text{where } \rho_{k-1,T} = \mathcal{L}_{k-1,T}\mathcal{L}_{k-2,T} \cdots \mathcal{L}_{1,T}\rho^{\mathrm{i}}.$$

- $\sigma_{k,T}$ depends on parameters at step k and $\{\mathcal{L}_{j,T}\}_{j=0}^{k}$

Strategy

- Our key object:

$$\sigma_{k,T} = S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big),$$
$$\text{where } \rho_{k-1,T} = \mathcal{L}_{k-1,T}\mathcal{L}_{k-2,T} \cdots \mathcal{L}_{1,T}\rho^{\mathrm{i}}.$$

- $\sigma_{k,T}$ depends on parameters at step k and $\{\mathcal{L}_{j,T}\}_{j=0}^{k}$
- Assume ρ^{i} is an eigenstate of $\mathcal{L}_{0,T}$. Then our adiabatic intuition tells us that for T large, $\rho_{1,T} = \mathcal{L}_{1,T}(\rho^{\mathrm{i}})$ is close to an eigenstate of $\mathcal{L}_{1,T}$.

Strategy

- Our key object:

$$\sigma_{k,T} = S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big),$$
$$\text{where } \rho_{k-1,T} = \mathcal{L}_{k-1,T}\mathcal{L}_{k-2,T}\cdots\mathcal{L}_{1,T}\rho^{\mathrm{i}}.$$

- $\sigma_{k,T}$ depends on parameters at step k and $\{\mathcal{L}_{j,T}\}_{j=0}^{k}$
- Assume ρ^{i} is an eigenstate of $\mathcal{L}_{0,T}$. Then our adiabatic intuition tells us that for T large, $\rho_{1,T} = \mathcal{L}_{1,T}(\rho^{\mathrm{i}})$ is close to an eigenstate of $\mathcal{L}_{1,T}$.
- Repeating this, we hope that adiabatically, $\rho_{k-1,T}$ is close to $\rho_{k,T}$, which we hope is close to an eigenstate of $\mathcal{L}_{k,T}$.

Strategy

- Our key object:

$$\sigma_{k,T} = S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big),$$
$$\text{where } \rho_{k-1,T} = \mathcal{L}_{k-1,T}\mathcal{L}_{k-2,T}\cdots\mathcal{L}_{1,T}\rho^{\mathrm{i}}.$$

- $\sigma_{k,T}$ depends on parameters at step k and $\{\mathcal{L}_{j,T}\}_{j=0}^{k}$
- Assume ρ^{i} is an eigenstate of $\mathcal{L}_{0,T}$. Then our adiabatic intuition tells us that for T large, $\rho_{1,T} = \mathcal{L}_{1,T}(\rho^{\mathrm{i}})$ is close to an eigenstate of $\mathcal{L}_{1,T}$.
- Repeating this, we hope that adiabatically, $\rho_{k-1,T}$ is close to $\rho_{k,T}$, which we hope is close to an eigenstate of $\mathcal{L}_{k,T}$.
- Then $\sigma_{k,T}$ approximately only depends on parameters at step k, and *not* on steps $0, \ldots, k-1$.

Outline

Outline

Three ingredients
1. Landauer's Principle
2. Adiabatic theorems
3. Repeated Interaction Systems

Combining the ingredients

Two tools
1. An adiabatic theorem for RIS
2. Perturbation of relative entropy

Entropy production of RIS in adiabatic limit

Two assumptions

Assume we have an adiabatic RIS, i.e. parameters are sampled from C^2 functions, yielding reduced dynamics $[0,1] \ni s \mapsto \mathcal{L}(s)$ and

Two assumptions

Assume we have an adiabatic RIS, i.e. parameters are sampled from C^2 functions, yielding reduced dynamics $[0,1] \ni s \mapsto \mathcal{L}(s)$ and

A1 $\mathcal{L}(s)$ is irreducible for each $s \in [0,1]$. That is, if P is a Hermitian projector such that $\mathcal{L}(P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P) \subset P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P$ then $P \in \{0, \mathrm{Id}\}$.

Two assumptions

Assume we have an adiabatic RIS, i.e. parameters are sampled from C^2 functions, yielding reduced dynamics $[0,1] \ni s \mapsto \mathcal{L}(s)$ and

A1 $\mathcal{L}(s)$ is irreducible for each $s \in [0,1]$. That is, if P is a Hermitian projector such that $\mathcal{L}(P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P) \subset P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P$ then $P \in \{0, \mathrm{Id}\}$.

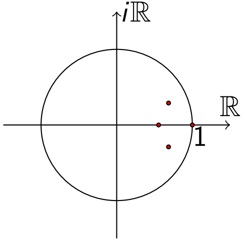
We know $\|\mathcal{L}(s)\| = 1$, so all eigenvalues are on or within the unit circle. Let $P(s)$ denote the eigenprojection onto eigenvalues on the unit circle.

Two assumptions

Assume we have an adiabatic RIS, i.e. parameters are sampled from C^2 functions, yielding reduced dynamics $[0,1] \ni s \mapsto \mathcal{L}(s)$ and

A1 $\mathcal{L}(s)$ is irreducible for each $s \in [0,1]$. That is, if P is a Hermitian projector such that $\mathcal{L}(P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P) \subset P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P$ then $P \in \{0, \mathrm{Id}\}$.

We know $\|\mathcal{L}(s)\| = 1$, so all eigenvalues are on or within the unit circle. Let $P(s)$ denote the eigenprojection onto eigenvalues on the unit circle.



Two assumptions

Assume we have an adiabatic RIS, i.e. parameters are sampled from C^2 functions, yielding reduced dynamics $[0,1] \ni s \mapsto \mathcal{L}(s)$ and

A1 $\mathcal{L}(s)$ is irreducible for each $s \in [0,1]$. That is, if P is a Hermitian projector such that $\mathcal{L}(P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P) \subset P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P$ then $P \in \{0, \mathrm{Id}\}$.

We know $\|\mathcal{L}(s)\| = 1$, so all eigenvalues are on or within the unit circle. Let $P(s)$ denote the eigenprojection onto eigenvalues on the unit circle.

A2 There is a uniform bound on the strictly contracting part of $\mathcal{L}(s)$, i.e. if $Q(s) := \mathrm{Id} - P(s)$,

$$\ell := \sup_{s\in[0,1]} \|\mathcal{L}(s)Q(s)\| < 1.$$

Two assumptions

Assume we have an adiabatic RIS, i.e. parameters are sampled from C^2 functions, yielding reduced dynamics $[0,1] \ni s \mapsto \mathcal{L}(s)$ and

ergodicity

- **A1** $\mathcal{L}(s)$ is irreducible for each $s \in [0,1]$. That is, if P is a Hermitian projector such that $\mathcal{L}(P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P) \subset P\mathcal{I}_1(\mathcal{H}_\mathcal{S})P$ then $P \in \{0, \mathrm{Id}\}$.

We know $\|\mathcal{L}(s)\| = 1$, so all eigenvalues are on or within the unit circle. Let $P(s)$ denote the eigenprojection onto eigenvalues on the unit circle.

- **A2** There is a uniform bound on the strictly contracting part of $\mathcal{L}(s)$, i.e. if $Q(s) := \mathrm{Id} - P(s)$,

$$\ell := \sup_{s\in[0,1]} \|\mathcal{L}(s)Q(s)\| < 1.$$

hard to check

Implications of these assumptions

1. At each step $1 \leq k \leq T$, $\mathcal{L}_{k,T} := \mathcal{L}(k/T)$ has a unique invariant state $\rho^{\text{inv}}_{k,T} > 0$.

Implications of these assumptions

1. At each step $1 \leq k \leq T$, $\mathcal{L}_{k,T} := \mathcal{L}(k/T)$ has a unique invariant state $\rho^{\text{inv}}_{k,T} > 0$.

Notation: $P^j_{k,T}$ is the projection on to the jth peripheral eigenvalue at step k, with adiabatic parameter T.

Implications of these assumptions

1. At each step $1 \leq k \leq T$, $\mathcal{L}_{k,T} := \mathcal{L}(k/T)$ has a unique invariant state $\rho^{\text{inv}}_{k,T} > 0$.

Notation: $P^j_{k,T}$ is the projection on to the jth peripheral eigenvalue at step k, with adiabatic parameter T.

2. The theorem: there exist constants $T_0 > 0$ and $C > 0$ such that for all $T > T_0$, there exists maps $(A_{k,T})_{k=1}^T$ such that

$$\|\mathcal{L}_{k,T} \cdots \mathcal{L}_{1,T} P_0 - A_{k,T}\| \leq \frac{C}{T(1-\ell)},$$
$$A_{k,T} P^j_0 = P^j_{k,T} A_{k,T}, \qquad A_{k,T} Q_0 = 0.$$

Implications of these assumptions

1. At each step $1 \leq k \leq T$, $\mathcal{L}_{k,T} := \mathcal{L}(k/T)$ has a unique invariant state $\rho^{\text{inv}}_{k,T} > 0$.

Notation: $P^j_{k,T}$ is the projection on to the jth peripheral eigenvalue at step k, with adiabatic parameter T.

2. The theorem: there exist constants $T_0 > 0$ and $C > 0$ such that for all $T > T_0$, there exists maps $(A_{k,T})_{k=1}^{T}$ such that

$$\|\mathcal{L}_{k,T} \cdots \mathcal{L}_{1,T} - A_{k,T}\| \leq \frac{C}{T(1-\ell)} + 2\ell^k,$$

$$A_{k,T} P^j_0 = P^j_{k,T} A_{k,T}, \qquad A_{k,T} Q_0 = 0.$$

Relaxation of these assumptions

Let $(X, \|\cdot\|)$ be a Banach space, $[0,1] \ni s \mapsto \mathcal{L}(s) \in \mathcal{B}(X)$ satisfy

H1. For each $s \in [0,1]$, $\mathcal{L}(s)$ is a contraction, i.e. $\|\mathcal{L}(s)\| \leq 1$.

Relaxation of these assumptions

Let $(X, \|\cdot\|)$ be a Banach space, $[0,1] \ni s \mapsto \mathcal{L}(s) \in \mathcal{B}(X)$ satisfy

H1. For each $s \in [0,1]$, $\mathcal{L}(s)$ is a contraction, i.e. $\|\mathcal{L}(s)\| \leq 1$.

H2. There is a uniform gap $\epsilon > 0$ such that for $s \in [0,1]$, $|e^j(s) - e^i(s)| > 2\epsilon$ for any $e^j(s) \neq e^i(s)$ in $\mathrm{sp}\,\mathcal{L}(s) \cap S^1$.

Relaxation of these assumptions

Let $(X, \|\cdot\|)$ be a Banach space, $[0,1] \ni s \mapsto \mathcal{L}(s) \in \mathcal{B}(X)$ satisfy

H1. For each $s \in [0,1]$, $\mathcal{L}(s)$ is a contraction, i.e. $\|\mathcal{L}(s)\| \leq 1$.

H2. There is a uniform gap $\epsilon > 0$ such that for $s \in [0,1]$, $|e^j(s) - e^i(s)| > 2\epsilon$ for any $e^j(s) \neq e^i(s)$ in $\operatorname{sp}\mathcal{L}(s) \cap S^1$.

H3. Let $P^j(s)$ be the spectral projector associated with $e^j(s) \in \operatorname{sp}\mathcal{L}(s) \cap S^1$, and $P(s) = \sum_j P^j(s)$ the *peripheral spectral projector*. The map $s \mapsto \mathcal{L}^P(s) := \mathcal{L}(s)P(s)$ is C^2 on $[0,1]$.

Relaxation of these assumptions

Let $(X, \|\cdot\|)$ be a Banach space, $[0,1] \ni s \mapsto \mathcal{L}(s) \in \mathcal{B}(X)$ satisfy

H1. For each $s \in [0,1]$, $\mathcal{L}(s)$ is a contraction, i.e. $\|\mathcal{L}(s)\| \leq 1$.

H2. There is a uniform gap $\epsilon > 0$ such that for $s \in [0,1]$, $|e^j(s) - e^i(s)| > 2\epsilon$ for any $e^j(s) \neq e^i(s)$ in $\operatorname{sp}\mathcal{L}(s) \cap S^1$.

H3. Let $P^j(s)$ be the spectral projector associated with $e^j(s) \in \operatorname{sp}\mathcal{L}(s) \cap S^1$, and $P(s) = \sum_j P^j(s)$ the *peripheral spectral projector*. The map $s \mapsto \mathcal{L}^P(s) := \mathcal{L}(s)P(s)$ is C^2 on $[0,1]$.

H4. There is a uniform bound on the strictly contracting part of $\mathcal{L}(s)$, i.e. if $Q(s) := \mathrm{Id} - P(s)$,

$$\ell := \sup_{s\in[0,1]} \|\mathcal{L}(s)Q(s)\| < 1.$$

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

1. Combinatorics. Approximate

$$\|\mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_1 P_0 - \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0\| = O(1/T).$$

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

1. Combinatorics. Approximate

$$\|\mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_1 P_0 - \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0\| = O(1/T).$$

   1.1 Expand $\prod_j (\mathcal{L}_j^P + \mathcal{L}_j^Q)$ and use that $Q_j P_{j-1}, P_j Q_{j-1} = O(1/T)$.

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

1. Combinatorics. Approximate
$$\|\mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_1 P_0 - \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0\| = O(1/T).$$
   1.1 Expand $\prod_j (\mathcal{L}_j^P + \mathcal{L}_j^Q)$ and use that $Q_j P_{j-1}, P_j Q_{j-1} = O(1/T)$.
   1.2 Carefully count. Sum terms with a geometric series using H4.

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

1. Combinatorics. Approximate
$$\|\mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_1 P_0 - \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0\| = O(1/T).$$
   1.1 Expand $\prod_j (\mathcal{L}_j^P + \mathcal{L}_j^Q)$ and use that $Q_j P_{j-1}, P_j Q_{j-1} = O(1/T)$.
   1.2 Carefully count. Sum terms with a geometric series using H4.
2. Follow a proof for the unitary case [Tan11]. Approximate
$$\|\mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0 - A_k\| = O(1/T).$$

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

1. Combinatorics. Approximate
$$\|\mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_1 P_0 - \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0\| = O(1/T).$$
   1.1 Expand $\prod_j (\mathcal{L}_j^P + \mathcal{L}_j^Q)$ and use that $Q_j P_{j-1}, P_j Q_{j-1} = O(1/T)$.
   1.2 Carefully count. Sum terms with a geometric series using H4.
2. Follow a proof for the unitary case [Tan11]. Approximate
$$\|\mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0 - A_k\| = O(1/T).$$
   2.1 Define Kato operators W_{j+1} to map from spectral subspaces at step j to spectral subspaces at time $j+1$.

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

1. Combinatorics. Approximate
$$\|\mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_1 P_0 - \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0\| = O(1/T).$$
   1.1 Expand $\prod_j (\mathcal{L}_j^P + \mathcal{L}_j^Q)$ and use that $Q_j P_{j-1}, P_j Q_{j-1} = O(1/T)$.
   1.2 Carefully count. Sum terms with a geometric series using H4.
2. Follow a proof for the unitary case [Tan11]. Approximate
$$\|\mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0 - A_k\| = O(1/T).$$
   2.1 Define Kato operators W_{j+1} to map from spectral subspaces at step j to spectral subspaces at time $j+1$.
   2.2 Define $A_k = \Phi W_k \cdots W_1$ where Φ is an appropriate dynamical phase.

Outline of proof

Set $\mathcal{L}_k^P := \mathcal{L}_k P_k$ and $\mathcal{L}_k^Q := \mathcal{L}_k Q_k$ then $\mathcal{L}_k = \mathcal{L}_k^P + \mathcal{L}_k^Q$.

1. Combinatorics. Approximate

$$\|\mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_1 P_0 - \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0\| = O(1/T).$$

   1.1 Expand $\prod_j (\mathcal{L}_j^P + \mathcal{L}_j^Q)$ and use that $Q_j P_{j-1}, P_j Q_{j-1} = O(1/T)$.
   1.2 Carefully count. Sum terms with a geometric series using H4.

2. Follow a proof for the unitary case [Tan11]. Approximate

$$\|\mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0 - A_k\| = O(1/T).$$

   2.1 Define Kato operators W_{j+1} to map from spectral subspaces at step j to spectral subspaces at time $j+1$.
   2.2 Define $A_k = \Phi W_k \cdots W_1$ where Φ is an appropriate dynamical phase.
   2.3 Rearrange, sum by parts, and use $s \mapsto P(s)$ is C^2 to bound a second discrete derivative.

Outline

Perturbation theory applied to relative entropy

For a faithful state η and small enough traceless perturbations D_1, D_2,

$$S(\eta + D_1 | \eta + D_2) = O((D_1 - D_2)^2).$$

Perturbation theory applied to relative entropy

For a faithful state η and small enough traceless perturbations D_1, D_2,

$$S(\eta + D_1 | \eta + D_2) = O((D_1 - D_2)^2).$$

In fact, if $\eta = \sum_i \mu_i p_i$ is the eigendecomposition of η, then

$$\left| S(\eta + D_1 | \eta + D_2) - F_\eta(D_1 - D_2) \right| = O((\|D_1\| + \|D_2\|)^3)$$

where $F_\eta(A) := F_\eta(A, A)$ for

$$F_\eta(A, B) := \sum_i \mathrm{Tr}(A p_i B p_i) \frac{1}{2\mu_i} + \sum_{i<j} \mathrm{Tr}(A p_j B p_i) \frac{\log(\mu_i) - \log(\mu_j)}{\mu_i - \mu_j}.$$

Perturbation theory applied to relative entropy

For a faithful state η and small enough traceless perturbations D_1, D_2,

$$S(\eta + D_1|\eta + D_2) = O((D_1 - D_2)^2).$$

In fact, if $\eta = \sum_i \mu_i p_i$ is the eigendecomposition of η, then

$$\left|S(\eta + D_1|\eta + D_2) - F_\eta(D_1 - D_2)\right| = O((\|D_1\| + \|D_2\|)^3)$$

where $F_\eta(A) := F_\eta(A, A)$ for

$$\boxed{F_\eta(A, B)} := \sum_i \mathrm{Tr}(Ap_iBp_i)\frac{1}{2\mu_i} + \sum_{i<j} \mathrm{Tr}(Ap_jBp_i)\frac{\log(\mu_i) - \log(\mu_j)}{\mu_i - \mu_j}.$$

2nd order term: $\|F_\eta(A)\| = O(\|A\|^2)$.

Outline

Perturbing $\sigma_{k,T}$

Now we want to write $\sigma_{k,T} = S(\eta + D_1 | \eta + D_2)$. But

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big).$$

Perturbing $\sigma_{k,T}$

Now we want to write $\sigma_{k,T} = S(\eta + D_1|\eta + D_2)$. But

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T}|\mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big).$$

So we want to approximate $\rho_{k-1,T} \approx \rho_{k,T} \approx \rho_k^{\mathrm{inv}}$.

Perturbing $\sigma_{k,T}$

Now we want to write $\sigma_{k,T} = S(\eta + D_1 | \eta + D_2)$. But

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big).$$

So we want to approximate $\rho_{k-1,T} \approx \rho_{k,T} \approx \rho^{\mathrm{inv}}_k$. The error is the difference

$$D_{k,T} := \mathcal{L}_{k,T}(\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T} - U_{k,T}\big((\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big) U^*_{k,T}.$$

Perturbing $\sigma_{k,T}$

Now we want to write $\sigma_{k,T} = S(\eta + D_1|\eta + D_2)$. But

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T}|\mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big).$$

So we want to approximate $\rho_{k-1,T} \approx \rho_{k,T} \approx \rho_k^{\mathrm{inv}}$. The error is the difference

$$D_{k,T} := \mathcal{L}_{k,T}(\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T} - U_{k,T}\big((\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big)U^*_{k,T}.$$

Next, we'll try to approximate $U_{k,T}\big((\rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big)U^*_{k,T} \approx \rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}$.

Perturbing $\sigma_{k,T}$

Now we want to write $\sigma_{k,T} = S(\eta + D_1|\eta + D_2)$. But

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T}|\mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big).$$

So we want to approximate $\rho_{k-1,T} \approx \rho_{k,T} \approx \rho^{\mathrm{inv}}_k$. The error is the difference

$$D_{k,T} := \mathcal{L}_{k,T}(\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T} - U_{k,T}\big((\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big)U^*_{k,T}.$$

Next, we'll try to approximate $U_{k,T}\big((\rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big)U^*_{k,T} \approx \rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}$. So the error is

$$X_{k,T} := U_{k,T}\, \rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}\, U^*_{k,T} - \rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}.$$

Perturbing $\sigma_{k,T}$

Now we want to write $\sigma_{k,T} = S(\eta + D_1 | \eta + D_2)$. But

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big).$$

So we want to approximate $\rho_{k-1,T} \approx \rho_{k,T} \approx \rho_k^{\mathrm{inv}}$. The error is the difference

$$D_{k,T} := \mathcal{L}_{k,T}(\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T} - U_{k,T}\big((\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big)U^*_{k,T}.$$

Next, we'll try to approximate $U_{k,T}\big((\rho^{\mathrm{inv}}_{k,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big)U^*_{k,T} \approx \rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}$. So the error is

$$X_{k,T} := U_{k,T}\,\rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}\,U^*_{k,T} - \rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}.$$

Using our adiabatic theorem and perturbation of entropy, we can find

$$\sigma_{k,T} = F_{\rho^{\mathrm{inv}}_{k,T} \otimes \xi^{\mathrm{i}}_{k,T}}(D_{k,T} - X_{k,T}) + O(1/T^3).$$

Back to our examples

Consider our 2x2 examples.

- With v_{RW}, $X_{k,T} \equiv 0$. Then

$$\sigma_{k,T} = F_{\rho_{k,T}^{\mathrm{inv}}}(D_{k,T}) + O(1/T^3) = O(1/T^2)$$

and the entropy production $\sigma_T^{\mathrm{tot}} \to 0$.

Back to our examples

Consider our 2x2 examples.

- With v_{RW}, $X_{k,T} \equiv 0$. Then

$$\sigma_{k,T} = F_{\rho_{k,T}^{\mathrm{inv}}}(D_{k,T}) + O(1/T^3) = O(1/T^2)$$

  and the entropy production $\sigma_T^{\mathrm{tot}} \to 0$.
- With v_{FD}, we find $X_{k,T} = O(\lambda) \neq 0$, and

$$\sigma_T^{\mathrm{tot}} = \sum_{k=1}^{T} F_{\rho_{k,T}^{\mathrm{inv}}}(X_{k,T} - D_{k,T}) + O(1/T^3) \sim \lambda^2 T \to \infty$$

Total entropy production

With the help of the lower bound

$$S(\rho|\eta) \geq \frac{1}{2}(\mathrm{Tr}\,|\rho - \eta|)^2$$

we have the following result.

Total entropy production

With the help of the lower bound

$$S(\rho|\eta) \geq \frac{1}{2}(\mathrm{Tr}\,|\rho - \eta|)^2$$

we have the following result.

Corollary

Denote by ρ_s^{inv} the unique invariant state for $\mathcal{L}(s)$, and let $X(s) = U(s)\big(\rho_s^{\mathrm{inv}} \otimes \xi_s\big)U(s)^* - \rho_s^{\mathrm{inv}} \otimes \xi_s$.

Total entropy production

With the help of the lower bound

$$S(\rho|\eta) \geq \frac{1}{2}(\operatorname{Tr}|\rho - \eta|)^2$$

we have the following result.

Corollary

Denote by ρ_s^{inv} the unique invariant state for $\mathcal{L}(s)$, and let $X(s) = U(s)\big(\rho_s^{\text{inv}} \otimes \xi_s\big)U(s)^* - \rho_s^{\text{inv}} \otimes \xi_s$.

- *If $X(s) = 0$ for all $s \in [0,1]$, and $\rho^{\text{i}} = \rho_{0,T}^{\text{inv}}$, then $\lim_{T\to\infty} \sigma_T^{tot} = 0$.*

Total entropy production

With the help of the lower bound

$$S(\rho|\eta) \geq \frac{1}{2}(\mathrm{Tr}\,|\rho - \eta|)^2$$

we have the following result.

Corollary

Denote by ρ_s^{inv} the unique invariant state for $\mathcal{L}(s)$, and let $X(s) = U(s)\big(\rho_s^{\mathrm{inv}} \otimes \xi_s\big)U(s)^* - \rho_s^{\mathrm{inv}} \otimes \xi_s$.

- *If $X(s) = 0$ for all $s \in [0,1]$, and $\rho^{\mathrm{i}} = \rho_{0,T}^{\mathrm{inv}}$, then $\lim_{T\to\infty} \sigma_T^{tot} = 0$.*
- *If $X(s) = 0$ for all $s \in [0,1]$, but $\rho^{\mathrm{i}} \neq \rho_{0,T}^{\mathrm{inv}}$, then $\lim_{T\to\infty} \sigma_T^{tot} < \infty$, but it is possibly non-zero.*

Total entropy production

With the help of the lower bound

$$S(\rho|\eta) \geq \frac{1}{2}(\mathrm{Tr}\,|\rho-\eta|)^2$$

we have the following result.

Corollary

Denote by ρ_s^{inv} *the unique invariant state for* $\mathcal{L}(s)$, *and let* $X(s) = U(s)\big(\rho_s^{\mathrm{inv}} \otimes \xi_s\big)U(s)^* - \rho_s^{\mathrm{inv}} \otimes \xi_s$.

- *If* $X(s) = 0$ *for all* $s \in [0,1]$, *and* $\rho^{\mathrm{i}} = \rho_{0,T}^{\mathrm{inv}}$, *then* $\lim_{T\to\infty} \sigma_T^{tot} = 0$.
- *If* $X(s) = 0$ *for all* $s \in [0,1]$, *but* $\rho^{\mathrm{i}} \neq \rho_{0,T}^{\mathrm{inv}}$, *then* $\lim_{T\to\infty} \sigma_T^{tot} < \infty$, *but it is possibly non-zero.*
- *If* $\sup_{s\in[0,1]} \|X(s)\|_1 > 0$, *then* $\sigma_T^{tot} \to \infty$ *in the limit* $T \to \infty$.

Detailed balance

Lemma

The following are equivalent:

1. $X(s) = 0$,

Detailed balance

Lemma

The following are equivalent:

1. $X(s) = 0$,
2. *there exists a self-adjoint operator* $K_{\mathcal{S}}(s)$ *on* $\mathcal{H}_{\mathcal{S}}$ *such that* $[K_{\mathcal{S}}(s) + h_{\mathcal{E}}(s), U(s)] = 0$.

Detailed balance

Lemma

The following are equivalent:

1. $X(s) = 0$,
2. *there exists a self-adjoint operator* $K_{\mathcal{S}}(s)$ *on* $\mathcal{H}_{\mathcal{S}}$ *such that* $[K_{\mathcal{S}}(s) + h_{\mathcal{E}}(s), U(s)] = 0$.

Moreover, if either condition holds we have the detailed balance relation that for all $\rho \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$,

$$\tilde{\mathcal{L}}(s)(\rho) = \mathrm{Tr}_{\mathcal{E}}\left(U(s)^*(\rho \otimes \xi_s)\, U(s)\right),$$

where

$$\tilde{\mathcal{L}}(s)(\rho) := (\rho_s^{\mathrm{inv}})^{+1/2}\mathcal{L}(s)^*\left((\rho_s^{\mathrm{inv}})^{-1/2}\rho\,(\rho_s^{\mathrm{inv}})^{-1/2}\right)(\rho_s^{\mathrm{inv}})^{+1/2}.$$

References

[JP14] Vojkan Jakšić and Claude-Alain Pillet. "A note on the Landauer principle in quantum statistical mechanics". In: *J. Math. Phys.* 55.7 (2014), 075210–75210:21.

[Kat50] Tosio Kato. "On the Adiabatic Theorem of Quantum Mechanics". In: *J. Phys. Soc. Japan* 5 (Nov. 1950), p. 435. DOI: 10.1143/JPSJ.5.435.

[Lan61] R. Landauer. "Irreversibility and Heat Generation in the Computing Process". In: *IBM Journal of Research and Development* 5.3 (1961), pp. 183–191.

[RW14] David Reeb and Michael M Wolf. "An improved Landauer principle with finite-size corrections". In: *New Journal of Physics* 16.10 (2014), p. 103011.

[Tan11] Atushi Tanaka. "Adiabatic Theorem for Discrete Time Evolution". In: *J. Phys. Soc. Japan* 80.12 (2011).

Outline

Three ingredients

1. Landauer's Principle
2. Adiabatic theorems
3. Repeated Interaction Systems

Combining the ingredients

Two tools

1. An adiabatic theorem for RIS
2. Perturbation of relative entropy

Entropy production of RIS in adiabatic limit