A local continuity bound for the entropy of finite distributions

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Entropy

Consider the finite set $\Omega = \{1, 2, \ldots, d\}$.

Let p be a probability distribution on Ω: $p(i) \geq 0$, $\sum_{i=1}^{d} p(i) = 1$.
We will call the set of all probability distributions on Ω by Δ.

Then the *entropy* of p is

$$H(p) := -\sum_{i=1}^{d} p(i) \log p(i).$$

- Measures information: how much to learn from observing an outcome
- Min: $(1, 0, \ldots, 0)$. Nothing to learn, zero entropy.
- Max: $(1/d, \ldots, 1/d)$. Most uncertainty.

Why entropy?

- Fundamental quantity in information theory
  - Shannon source coding. Given i.i.d. data distributed with probability distribution p, the asymptotic optimal lossless data-compression rate is $H(p)$.

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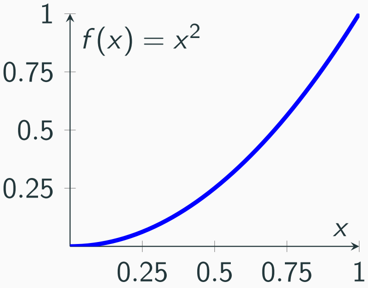
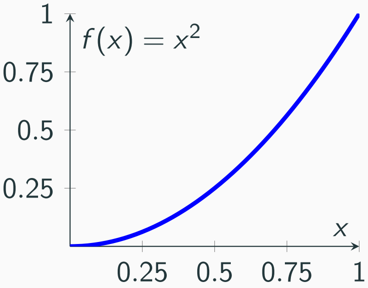
- Related to other quantities (e.g. conditional entropy when you don't condition on anything)
- The von Neumman entropy of a d-dimensional quantum state ρ is the (Shannon) entropy of its eigenvalues, which form a probability distribution on Ω.

Continuity bounds



Continuity bounds



Continuity bound: don't know the values x, y but know

- $x, y \in [0, 1]$
- $|x - y| \leq \varepsilon$.

Continuity bound:

$$|f(x) - f(y)| \ \leq \ 2\varepsilon$$

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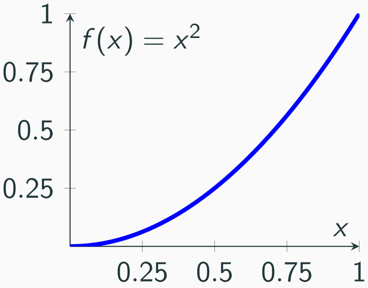

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Local continuity bound: do know the value x. LCB:

$$|f(x) - f(y)| \leq (x+1)\varepsilon.$$

Tighter bound for any $x < 1$.

Audenaert-Fannes bound

There already exists a continuity bound for entropy!

Lemma (Audenaert-Fannes)

Let $\varepsilon \in [0,1]$*, and* $p, q \in \Delta$ *such that* $\frac{1}{2}\|p-q\|_1 \leq \varepsilon$*. Then*

$$|H(p) - H(q)| \leq \begin{cases} \varepsilon \log(d-1) + h(\varepsilon) & \textit{if } \varepsilon < 1 - \frac{1}{d} \\ \log d & \textit{if } \varepsilon \geq 1 - \frac{1}{d}. \end{cases} \tag{AF}$$

LCB from maximal and minimal distributions

$$\boxed{B_\varepsilon(q) = \{p \in \Delta : \frac{1}{2}\|p - q\|_1 \leq \varepsilon\}}$$

Fix $q \in \Delta$ and $\varepsilon \in [0,1]$. We always have

$$H(p) - H(q) \leq \max_{p \in B_\varepsilon(q)} H(p) - H(q)$$

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$$|H(p) - H(q)| \le \max\Big\{ \max_{p \in B_\varepsilon(q)} H(p) - H(q), H(q) - \min_{p \in B_\varepsilon(q)} H(p)\Big\}$$

which is a function of q and ε only.

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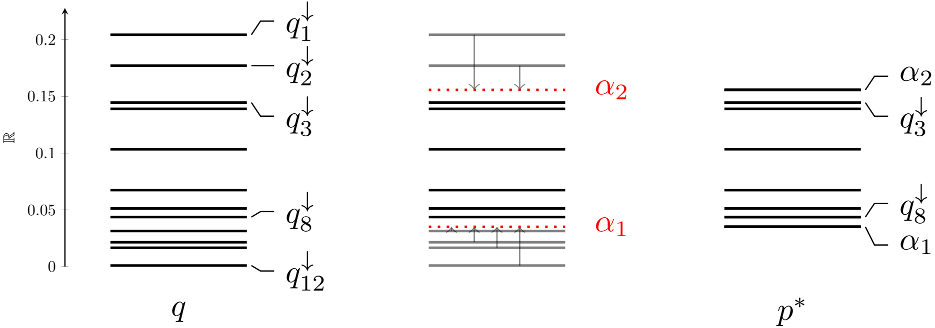
Idea: understand $p^* = \mathrm{argmax}_{p \in B_\varepsilon(q)} H(p)$ and $\mathrm{argmin}_{p \in B_\varepsilon(q)}(p)$.

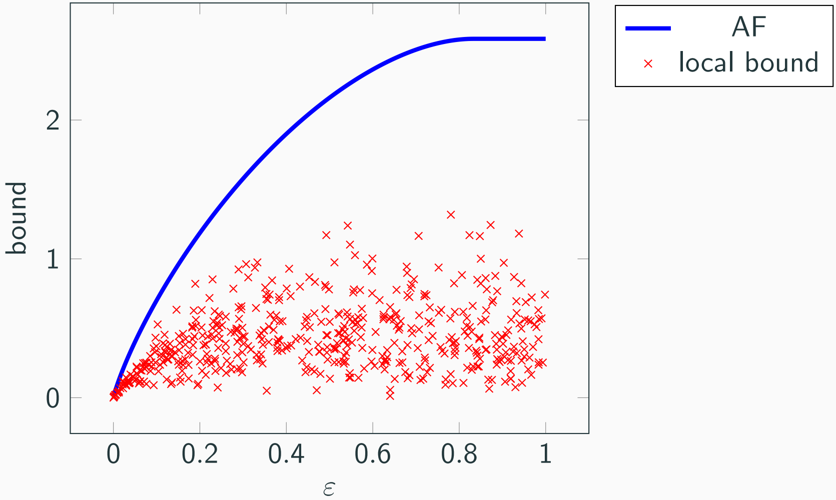
Convex optimization $\rightsquigarrow$ optimality conditions



$\mathbb{R}$
0.2
0.15
0.1
0.05
0
$q_1^{\downarrow}$
$q_2^{\downarrow}$
$q_3^{\downarrow}$
$q_8^{\downarrow}$
$q_{12}^{\downarrow}$
q
α_2
α_1
α_2
$q_3^{\downarrow}$
$q_8^{\downarrow}$
α_1
p^*

Comparison of local continuity bound and the AF bound

In red: 500 choices of $q \in \Delta$ and $\varepsilon \in [0, 1]$.



Thank you!