

Reducing to the classical case

$\mathcal{H}$: Hilbert space; $\dim \mathcal{H} = d < \infty$; $\mathcal{D} \equiv \mathcal{D}(\mathcal{H})$: set of density matrices;

$$B_\varepsilon(\sigma) := \{\rho \in \mathcal{D} : \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon\},$$

the ε -ball. Work in the basis where $\sigma = \text{Eig}^\downarrow(\sigma)$, where $\text{Eig}^\downarrow(\sigma)$ is the diagonal matrix of eigenvalues of σ in decreasing order. Then for $\rho \in B_\varepsilon(\sigma)$,

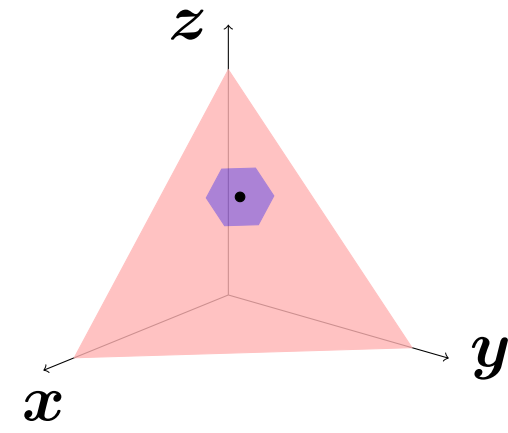
$$\frac{1}{2} \sum_{j=1}^d |\lambda_j^\downarrow(\rho) - \lambda_j^\downarrow(\sigma)| = \frac{1}{2} \|\text{Eig}^\downarrow(\rho) - \sigma\|_1 \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \varepsilon.$$

As $\text{Eig}^\downarrow(\rho)$ is unitarily equivalent to ρ , for single-partite entropies only need the ℓ^1 ball $B_\varepsilon(q) = \{p \in \text{Simp} : \frac{1}{2} \sum_j |p_j - q_j| \leq \varepsilon\}$, for $q = \vec{\lambda}^\downarrow(\sigma)$.

Right: q is the black dot, $B_\varepsilon(q)$ the purple hexagon, in dimensions $d = 3$.

$\lambda_j^\downarrow(\rho)$ denotes the j th largest eigenvalue of ρ .

$\text{Simp} := \{p \in \mathbb{R}^d : p_i \geq 0, \sum_i p_i = 1\}$.

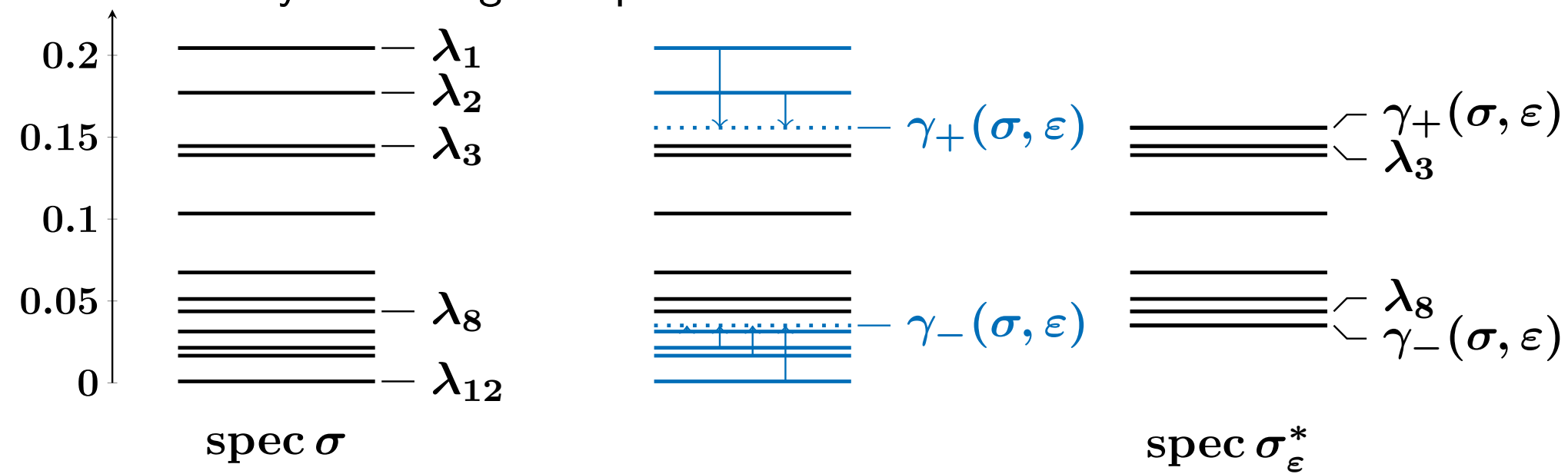


Local extrema in the majorization order

Given $\sigma \in \mathcal{D}$ and $\varepsilon \in (0, 1]$, we construct $\sigma_{*,\varepsilon}, \sigma_\varepsilon^* \in B_\varepsilon(\sigma)$ such that

$$\forall \omega \in B_\varepsilon(\sigma), \quad \sigma_\varepsilon^* \prec \omega \prec \sigma_{*,\varepsilon} \quad (1)$$

where $\prec$ is the majorization order (defined below). The state σ_ε^* is constructed by flattening the spectrum:



Specific case: pure state $\psi = \text{diag}(1, 0, \dots, 0)$; in the same basis we have

$$\psi_\varepsilon^* = \begin{cases} \text{diag}(1 - \varepsilon, \frac{\varepsilon}{d-1}, \dots, \frac{\varepsilon}{d-1}) & \varepsilon < 1 - \frac{1}{d} \\ \frac{1}{d} \mathbb{1} & \varepsilon \geq 1 - \frac{1}{d} \end{cases} \quad (2)$$

Eq. (1) yields a *local continuity bound* for H Schur concave: $\forall \omega \in B_\varepsilon(\sigma)$,

$$|H(\omega) - H(\sigma)| \leq \max\{H(\sigma_\varepsilon^*) - H(\sigma), H(\sigma) - H(\sigma_{*,\varepsilon})\}. \quad (3)$$

Uniform bounds from local majorization

For H Schur concave, $\rho, \sigma \in \mathcal{D}$ with $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$,

$$|H(\rho) - H(\sigma)| \leq \max\{H(\sigma_\varepsilon^*) - H(\sigma), H(\rho_\varepsilon^*) - H(\rho)\} \leq \sup_{\omega \in \mathcal{D}} \Delta_\varepsilon(\omega) \quad (4)$$

where $\Delta_\varepsilon : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$ defined by $\Delta_\varepsilon(\omega) := H(\omega_\varepsilon^*) - H(\omega)$.

Goal: maximize Δ_ε over $\mathcal{D}$ for some useful choices of H .

Definition of majorization

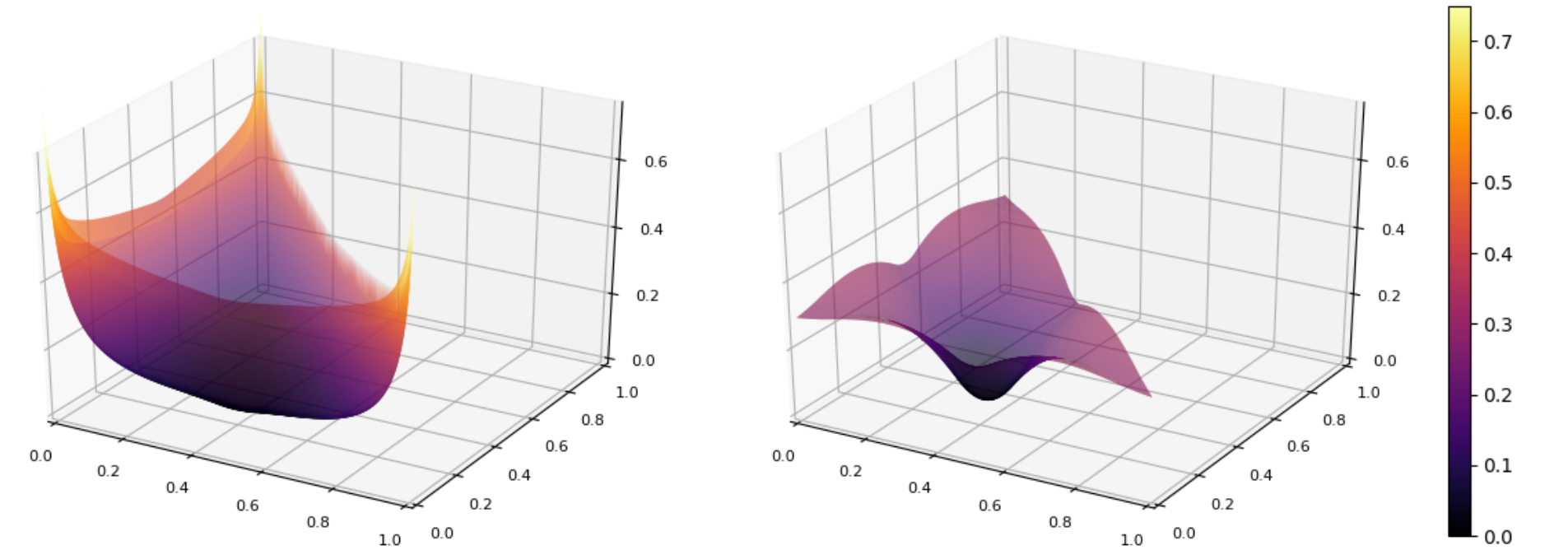
- ▶ We say $\rho \prec \sigma$ if the vector of eigenvalues $\vec{\lambda}^\downarrow(\rho) \prec \vec{\lambda}^\downarrow(\sigma)$.
- ▶ For $p \in \mathbb{R}^d$, set $p^\downarrow$ as the permutation of p in decreasing order. Then for $p, q \in \mathbb{R}^d$, we say $p \prec q$ if $\sum_{i=1}^k p^\downarrow(i) \leq \sum_{i=1}^k q^\downarrow(i)$ for each $k = 1, 2, \dots, d$, with equality for $k = d$.
- ▶ $H : \mathcal{D} \rightarrow \mathbb{R}$ is *Schur concave* if $\rho \prec \sigma \implies H(\rho) \leq H(\sigma)$.

Right: Comparison of continuity bounds for the Rényi entropy with $\alpha = \frac{1}{2}$ and $d = 5$. The small crosses are points corresponding to the right-hand side of (3) for 500 randomly chosen states σ .

- ▶ [Ras11]: A. E. Rastegin. (2011). arXiv: 1012.5356
- ▶ [Che+17]: Z. Chen et al. (2017). arXiv: 1701.02398

Plotting Δ_ε for α -Rényi entropies

Take $H(\rho) = S_\alpha(\rho) := \frac{1}{1-\alpha} \log[\text{Tr}(\rho^\alpha)]$, the α -Rényi entropy. In $d = 3$, we parametrize $\sigma = \text{diag}(x, y, 1 - x - y)$ and plot $(x, y) \mapsto \Delta_\varepsilon(\sigma)$. Left: $\alpha = 0.5$, right: $\alpha = 2.0$.



Observation: for $\alpha \leq 1$, the maxima occur at $(0, 0)$, $(1, 0)$, and $(0, 1)$ i.e. when σ is a pure state. For $\alpha > 1$, the maxima change with α .

Schur convexity of Δ_ε

Δ_ε is not convex even for the von Neumann entropy (counterexamples). However, Δ_ε is **Schur convex**, when $H = H_{(h,\phi)}$ for

$$H_{(h,\phi)}(\rho) := h(\text{Tr}[\phi(\rho)]) \quad (5)$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ is concave, $\phi : [0, 1] \rightarrow \mathbb{R}$ strictly concave, $\phi(0) = 0$, and $h(\phi(1)) = 0$. Examples include

- ▶ the von Neumann entropy $S(\rho) := -\text{Tr}[\rho \log \rho]$,
- ▶ the q -Tsallis entropy $T_q(\rho) := \frac{1}{1-q} [\text{Tr}(\rho^q) - 1]$ for $q \in (0, 1) \cup (1, \infty)$,
- ▶ the α -Rényi entropy $S_\alpha(\rho) := \frac{1}{1-\alpha} \log[\text{Tr}(\rho^\alpha)]$ for $\alpha \in (0, 1)$.

Brief proof sketch

$\mathcal{M}_\varepsilon(\omega) := \omega_\varepsilon^*$. **Semigroup property:** $\mathcal{M}_{\varepsilon_1+\varepsilon_2}(\omega) = \mathcal{M}_{\varepsilon_1}(\mathcal{M}_{\varepsilon_2}(\omega))$. Then $\Delta_{\varepsilon_1+\varepsilon_2}(\omega) = \Delta_{\varepsilon_1}(\mathcal{M}_{\varepsilon_2}(\omega)) + \Delta_{\varepsilon_2}(\omega)$. By iterating, only need to prove Schur convexity of Δ_ε for small ε , where $\mathcal{M}_\varepsilon(\sigma)$ only moves the smallest and largest eigenvalues of σ . Reduces to concavity of ϕ on $\mathbb{R}$.

Main result: tight uniform bounds

Schur convex on $\mathcal{D}$ implies maxima are pure states. By (4), if $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$,

$$|H_{(h,\phi)}(\rho) - H_{(h,\phi)}(\sigma)| \leq H_{(h,\phi)}(\psi_\varepsilon^*) \quad (6)$$

for any pure state ψ , where ψ_ε^* is given by (2). In particular, for $\alpha \in (0, 1)$,

$$|S_\alpha(\rho) - S_\alpha(\sigma)| \leq \frac{1}{1-\alpha} \log((1-\varepsilon)^\alpha + (d-1)^{1-\alpha} \varepsilon^\alpha) \quad (7)$$

for $\varepsilon < 1 - \frac{1}{d}$, and the trivial bound $\log d$ for $\varepsilon \geq 1 - \frac{1}{d}$, where $d = \dim \mathcal{H}$. Moreover, equality in (6) or (7) occurs if and only if one of the two states (say, σ) is pure, and $\rho = \sigma_\varepsilon^*$ given by (2).

Performance of the bounds by example

