Entanglement breaking times for quantum Markovian evolutions

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Overview

Qualitative level: what evolutions eventually break entanglement?

Quantitative level: when do evolutions eventually break entanglement?

Outline

Qualitative level: what evolutions eventually break entanglement?

Quantitative level: when do evolutions eventually break entanglement?

Asympt. convergence $\rightsquigarrow$ loss of entanglement

Let Φ_t be a linear map representing time evolution from time 0 to time t. If

$$\Phi_t(\rho) \xrightarrow{t\to\infty} \mathrm{tr}[\rho]\sigma \qquad \forall\, \rho$$

Then by linearity,

$$\Phi_t \otimes \mathrm{id}(\rho_{AB}) \xrightarrow{t\to\infty} \sigma \otimes \rho_B \qquad \forall\, \rho_{AB}.$$

Asympt. convergence $\rightsquigarrow$ loss of entanglement *in finite time*

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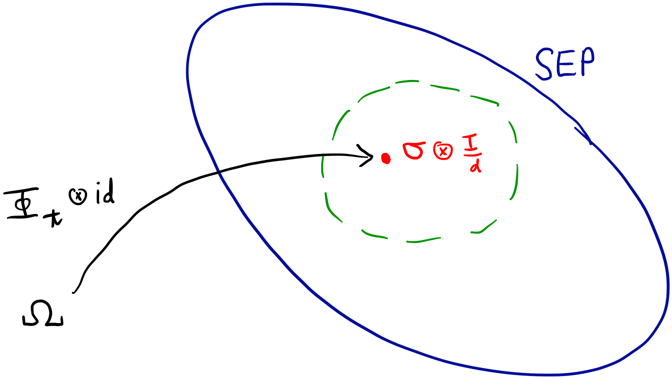


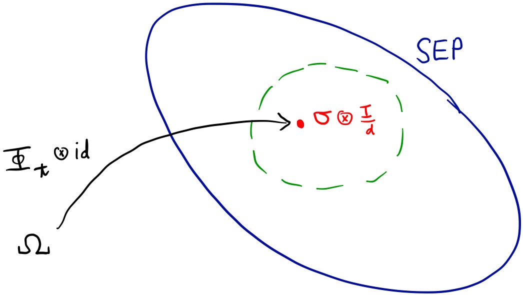
Open balls in the set of separable states

Theorem (Gurvits and Barnum, 2002)

$I \otimes I + \Delta$ is separable for any $\Delta = \Delta^*$ with $\|\Delta\|_2 \leq 1$.

Corollary (H., Rouzé, Stilck França '19)

$\omega \in \mathcal{D}(\mathcal{H})$ is separable whenever $\|\omega - \rho \otimes \sigma\|_2 \leq \lambda_{min}(\rho)\lambda_{min}(\sigma)$.



Characterize EEB in a simple case

If $(\Phi_t)_{t\geq 0}$ is a family of quantum channels such that for some $\sigma > 0$,

$$\Phi_t(\rho) \xrightarrow{t\to\infty} \sigma \qquad \forall\, \rho \tag{1}$$

then Φ_{t_0} is entanglement breaking for some $t_0 < \infty$. That is, $(\Phi_t)_{t\geq 0}$ *eventually entanglement breaking (EEB)*.

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In fact, if $(\Phi_t)_{t\geq 0}$ is additionally a continuous-time semigroup:

1. $\Phi_{s+t} = \Phi_s \circ \Phi_t$,
2. $\Phi_0 = \mathrm{id}$,
3. $t \to \Phi_t$ is continuous

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Then $(\Phi_t)_{t\geq 0}$ being EEB implies $\underbrace{\exists\, \sigma > 0 \text{ such that (1) holds}}_{\text{primitive}}$.

This follows from a result of [Lami and Giovannetti '16] using that $\det|\Phi_t| \neq 0$.

Can we reduce our assumptions?

- Continuous-time semigroup: $(\Phi_t)_{t\geq 0}$. To evolve from t to $t+\varepsilon$,
$$\Phi_t(\rho) \to \Phi_{t+\varepsilon}(\rho) = \Phi_\varepsilon(\Phi_t(\rho))$$
governed by Φ_ε alone: infinitesimally memoryless.
$(\Phi_t = \exp(t\mathcal{L}))$
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- Next step: Discrete-time semigroup: $\{\Phi^n\}_{n\in\mathbb{N}}$ has $\Phi^{n+m} = \Phi^n \circ \Phi^m$, where Φ is a quantum channel.
  - Φ itself could involve an arbitrary interaction with an environment (e.g. Stinespring), but to evolve for two steps, one uses a fresh environment (no memory shared between uses of Φ)

What is different in discrete-time? (mathematically)

- Continuous time: require $\Phi_0 = \mathrm{id}$ (important to obtain $\Phi_t = \exp(t\mathcal{L})$, etc.).
  - Analogy to discrete-time:

$$\lim_{s\to 0} \Phi^s \equiv \text{projection onto supp } \Phi = \mathrm{id}$$

  We do not require this! Can have $\det \Phi = 0$.

- Irreducibility: Φ (or Φ_t) is irreducible if only trivial blocks preserved, i.e.

$$\Phi(P\mathcal{B}(\mathcal{H})P) \subseteq P\mathcal{B}(\mathcal{H})P \implies P \in \{0, I\}$$

  for orthogonal projections P.
  - For continuous-time semigroups: primitive $\iff$ irreducible.
  - For discrete-time semigroups: primitive $\Rightarrow$ irreducible ($\not\Leftarrow$).

Goal: Characterize quantum channels Φ such that $\{\Phi^n\}_{n\in\mathbb{N}}$ is eventually entanglement breaking.

(Continuous-time semigroup: $(\Phi_t)_{t\geq 0}$ is eventually entanglement breaking if and only if it is primitive.)

Important special case: PPT channels

PPT state: state with positive partial transpose: $\rho_{AB}^{T_A} \geq 0$.
PPT map linear map Φ whose Choi state $J(\Phi)$ is PPT.

- Peres, 1996: Separable states are PPT; entangled states may not be
- Horodecki (M., P., R.), 1998: Maximally entangled qubits cannot be distilled from PPT states (i.e. they are *bound entangled*)
- Horodecki (K., M., P.), Oppenheim, 2005: There are PPT entangled states from which private keys can be distilled

Question: if Alice and Bob share a PPT entangled state, and Bob and Charlie share a PPT entangled state, can they perform a "repeater protocol" to distill a private key between Alice and Bob?

- PPT^2 conjecture: if Φ is a PPT map, then $\Phi \circ \Phi$ is EB.
- Christandl, Ferrara, 2017: PPT^2 conjecture $\implies$ answer is no

Asymptotics: What is known? (I)

[Lami and Giovannetti '16]: the limit points of the sequence $\{\Phi^n\}_{n\in\mathbb{N}}$ are

- either all entanglement breaking (Φ is *asymptotically entanglement breaking (AEB)*)
- or none of them are entanglement breaking (Φ is *asymptotically entanglement saving (AES)*)

Asymptotics: What is known? (II)

Φ: quantum channel.

(Dual of) decoherence-free algebra:

$$\tilde{\mathcal{N}}(\Phi) := \bigoplus_{i=1}^{K} \mathcal{B}(\mathcal{H}_i) \otimes \tau_i \oplus 0_{\mathcal{K}_0} \, .$$

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If $X \in \tilde{\mathcal{N}}(\Phi)$ then $X = \bigoplus_{i=1}^{K} X_i \otimes \tau_i \oplus 0_{\mathcal{K}_0}$ and

$$\Phi(X) = \bigoplus_{i=1}^{K} U_i X_{\pi(i)} U_i^{\dagger} \otimes \tau_i \oplus 0_{\mathcal{K}_0} \,.$$

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Theorem (Lami and Giovannetti '16)

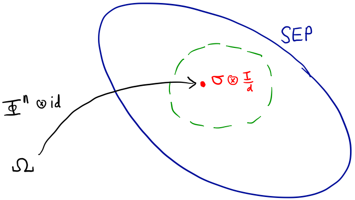
Φ *is asymptotically entanglement breaking iff* $d_i = 1\ \forall i$.

PPT maps: What is known?

- Kennedy, Manor, Paulsen, 2017
  - PPT quantum channels are AEB
- Rahaman, Jaques, Paulsen, 2018
  - unital PPT quantum channels are EEB
- Christandl, Müller-Hermes, Wolf 2018
  - PPT-square holds for $d_{\mathcal{H}} = 3$, and for Gaussian channels
- Collins, Yin, Zhong, 2018
  - PPT-square holds generically for a class of random maps

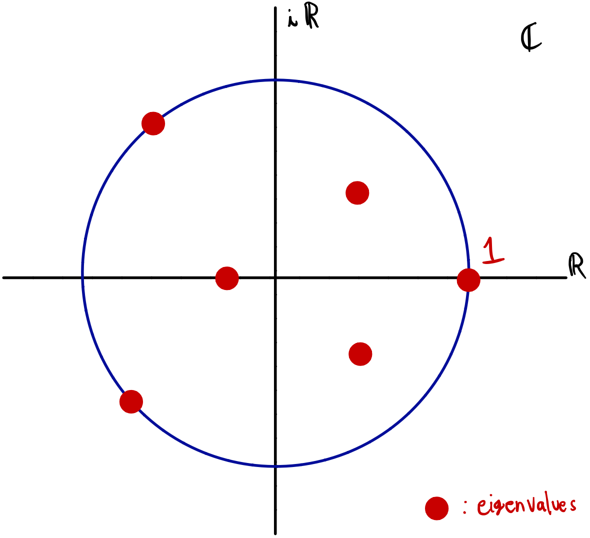
First step: irreducible quantum channels

- Irreducible: $\Phi(P\mathcal{B}(\mathcal{H})P) \subseteq P\mathcal{B}(\mathcal{H})P$ holds only for trivial P ($P = 0$ or $P = I$).
- Continuous-time: irreducible iff primitive iff EEB.
- Discrete-time: Primitive $\implies$ EEB



- Are irreducible channels in discrete time EEB?

Eigenvalues of quantum channels



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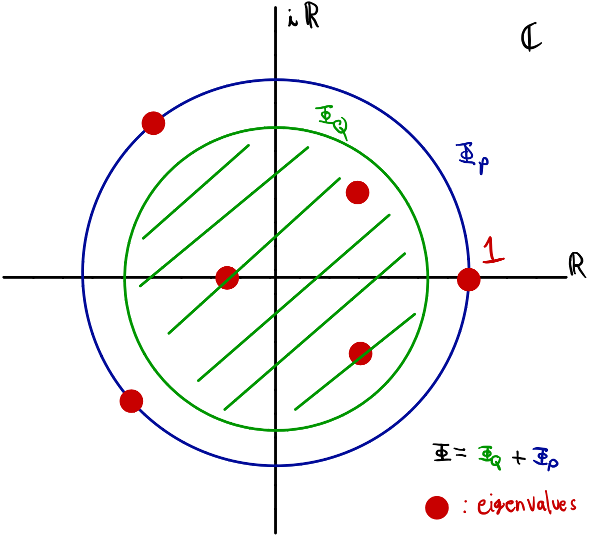

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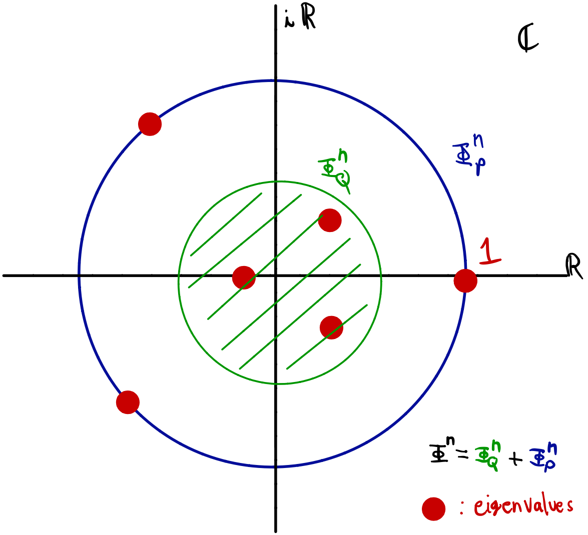


Irreducible quantum channels (Perron-Frobenius theory)

Φ: irreducible quantum channel

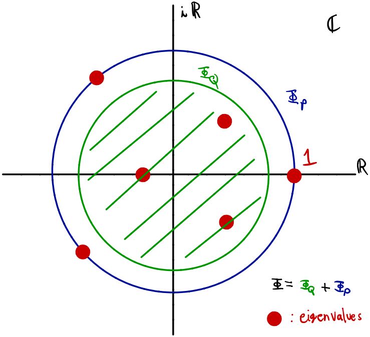
- Peripheral eigenvalues: zth roots of unity for some $z \in \mathbb{N}$ called the *period*

- Primitive iff $z = 1$.

- Unique invariant state σ (which is full rank: $\sigma > 0$)

- Perron-Frobenius projections: $\{p_j\}_{j=0}^{z-1}$ decomposition of $\mathcal{H}$,
  1. $\sum_j p_j = I$,
  2. $\Phi(p_j X p_k) = p_{j-1}\Phi(X)p_{k-1}$ intertwining (indicies taken mod z)
  3. $[p_j, \sigma] = 0 \qquad \forall j$

Irreducible quantum channels (II)



- $\Phi = \Phi_P + \Phi_Q$ with $\Phi_P \circ \Phi_Q = \Phi_Q \circ \Phi_P = 0$, $\Phi_Q^n \to 0$,
- Peripheral action fixed: $\Phi_P^n(X) = z \sum_{j=0}^{z-1} \mathrm{tr}[p_j X] p_{j-n}\sigma$,
- $J(\Phi_P^n) = \sum_{j=0}^{z-1} \mathrm{tr}[p_j] \frac{p_{j-n}\sigma p_{j-n}}{\mathrm{tr}[p_{j-n}\sigma]} \otimes \frac{p_j}{\mathrm{tr}[p_j]} \in \mathrm{SEP}$

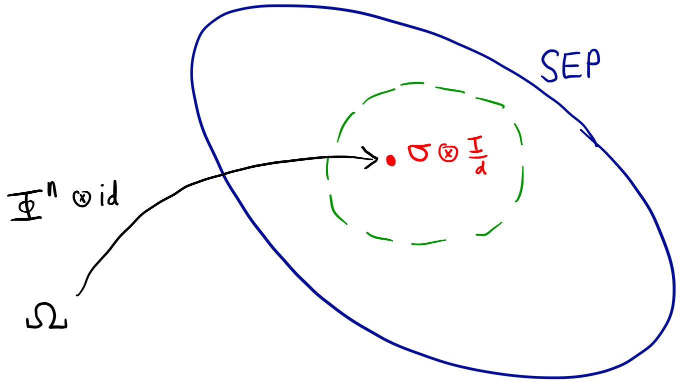
Are irreducible channels EEB? Perturbative argument fails

1. Rewrite:

$$J(\Phi_P^n) = zL_n(\sigma \otimes I)L_n, \qquad L_n := \sum_{j=0}^{z-1} p_{j-n} \otimes p_j$$

2. Note: $L_n = L_n^2 = L_n^*$, $\sum_{n=0}^{z-1} L_n = I \otimes I$.
3. So: $J(\Phi_P^n) \in L_n \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) L_n$ only lives in one part of the Hilbert space
4. Can construct entangled states Ω_k in $L_k \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) L_k$ for any k (if $z > 1$, i.e. non-primitive)
5. Then $\forall \varepsilon$, $\varepsilon \Omega_k + (1-\varepsilon) J(\Phi_P^n)$ is entangled (for $n \neq k \bmod z$)
6. $J(\Phi_P^n)$ is close to $J(\Phi^n)$ for large n
7. Conclusion: even for large n, there are entangled states close to $J(\Phi^n)$ if Φ is irreducible and non-primitive.

Compare to primitive:



This method fails for irreducible!
(In fact, there are non-EEB irreducible channels).

How to proceed? An intermediate step, PPT channels

Recall: a PPT quantum channel is one whose Choi matrix has a positive partial transpose:

$$(\mathcal{T} \otimes \mathrm{id})J(\Phi) \equiv \sum_{i,j} \Phi(|i\rangle\langle j|)^T \otimes |i\rangle\langle j| \geq 0$$

Also: If Φ is PPT, then $\Phi \otimes \mathrm{id}(\rho_{AB})$ is PPT for all ρ_{AB}.

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Goal:

$$\text{EEB} \implies \begin{array}{c}\text{eventually}\\ \text{PPT}\end{array} \implies \begin{array}{c}\text{eventually}\\ \bigoplus\text{-irreducible}\end{array} \implies \begin{array}{c}\text{eventually}\\ \bigoplus\text{-primitive}\end{array} \implies \text{EEB}$$

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Every separable state is PPT, so every entanglement-breaking channel is PPT.

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*under an assumption (faithfulness)

Key lemma: PPT kills off-diagonal blocks

Definition ($\{p_i\}_{i=0}^{z-1}$-block preserving)

Given an orthogonal resolution of the identity, $\{p_i\}_{i=0}^{z-1}$, we say that a quantum channel Φ is $\{p_i\}_{i=0}^{z-1}$-block preserving if for all $i, j \in \{0, \ldots, z-1\}$,

$$\Phi(p_i \mathcal{B}(\mathcal{H}) p_j) \subseteq p_i \mathcal{B}(\mathcal{H}) p_j.$$

Lemma (H., Rouzé, Stilck França '19)

If Φ is a PPT *quantum channel and is $\{p_i\}_{i=0}^{z-1}$-block preserving, then*

$$\Phi(p_i \mathcal{B}(\mathcal{H}) p_j) = \{0\}$$

for all $i \neq j$.

Proof by the contrapositive

Goal: $\boxed{\exists i \neq j \text{ s.t } \Phi(p_i\mathcal{B}(\mathcal{H})p_j) \neq \{0\} \quad \Longrightarrow \quad \Phi \text{ not PPT}}$

Assume $|0\rangle \in p_0\mathcal{H}$ and $|1\rangle \in p_1\mathcal{H}$ such that $\Phi(|0\rangle\langle 1|) \neq 0$. Let

$$|\Omega\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

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$$\Phi \otimes \mathrm{id}(\Omega) = \left(\begin{array}{cc|cc} \Phi(|0\rangle\langle 0|) & 0 & 0 & \Phi(|1\rangle\langle 0|) \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \Phi(|0\rangle\langle 1|) & 0 & 0 & \Phi(|1\rangle\langle 1|) \end{array}\right)$$

as $\{p_0, p_1\}$ blocks in the $\{|0\rangle, |1\rangle\}$ basis.

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That is,

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$$\mathcal{T} \circ \Phi \otimes \mathrm{id}(\Omega) = \left(\begin{array}{cc|cc} \Phi(|0\rangle\langle 0|) & 0 & 0 & 0 \\ 0 & 0 & \Phi(|1\rangle\langle 0|) & 0 \\ \hline 0 & \Phi(|0\rangle\langle 1|) & 0 & 0 \\ 0 & 0 & 0 & \Phi(|1\rangle\langle 1|) \end{array}\right)$$

as $\{p_0, p_1\}$ blocks in the $\{|0\rangle, |1\rangle\}$ basis.

That is,

$$\mathcal{T} \circ \Phi \otimes \mathrm{id}(\Omega) = \Phi(|0\rangle\langle 0|) \oplus \underbrace{\begin{pmatrix} 0 & \Phi(|0\rangle\langle 1|) \\ \Phi(|1\rangle\langle 0|) & 0 \end{pmatrix}}_{\text{Hermitian, traceless, non-zero } \Longrightarrow \not\geq 0} \oplus \Phi(|1\rangle\langle 1|)$$

not PPT!

Reduction to irreducible (I)

$$\text{EEB} \underset{\checkmark}{\Longrightarrow} \begin{array}{c}\text{eventually}\\ \text{PPT}\end{array} \underset{?}{\Longrightarrow} \begin{array}{c}\text{eventually}\\ \bigoplus\text{-irreducible}\end{array} \underset{?}{\Longrightarrow} \begin{array}{c}\text{eventually}\\ \bigoplus\text{-primitive}\end{array} \underset{\checkmark}{\Longrightarrow} \text{EEB}$$

Let Φ be a faithful quantum channel: $\exists\, \sigma > 0$ such that $\Phi(\sigma) = \sigma$.

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Then $\exists$ orthogonal projections $P_1, P_2, \ldots, P_N$ such that

1. $\sum_j P_j = I$,
2. $\Phi(P_j \mathcal{B}(\mathcal{H}) P_k) \subseteq P_j \Phi(\mathcal{B}(\mathcal{H})) P_k \qquad \forall j, k$
3. $\Phi|_{P_j \mathcal{B}(\mathcal{H}) P_j}$ is irreducible

Reduction to irreducible (II)

Let Φ be a faithful quantum channel. Then $\exists P_1, P_2, \ldots, P_N$ s.t.

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for all $m \geq d_{\mathcal{H}}^2$.

- Hence,

$$\Phi^m = \bigoplus_j \underbrace{(\Phi|_{P_j \mathcal{B}(\mathcal{H}) P_j})}_{\text{irreducible}}{}^m \qquad \forall m \geq d_{\mathcal{H}}^2$$

$$\text{EEB} \underset{\checkmark}{\Longrightarrow} \begin{array}{c}\text{eventually}\\ \text{PPT}\end{array} \underset{\color{red}?}{\Longrightarrow} \begin{array}{c}\text{eventually}\\ \bigoplus\text{-irreducible}\end{array} \underset{\color{red}?}{\Longrightarrow} \begin{array}{c}\text{eventually}\\ \bigoplus\text{-primitive}\end{array} \underset{\checkmark}{\Longrightarrow} \text{EEB}$$

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Let Φ be an irreducible EEB quantum channel.

- Then for some n, Φ^{nz} is PPT. Since

$$\Phi^{nz}(p_j X p_k) = p_{j-nz}\Phi^n(X)p_{k-nz} \subseteq p_j\mathcal{B}(\mathcal{H})p_k$$

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- So by the Key Lemma,

$$\Phi^{nz}(p_j\mathcal{B}(\mathcal{H})p_k) = \{0\} \qquad \forall j \neq k \tag{2}$$

  and hence $\Phi^z|_{p_j\mathcal{B}(\mathcal{H})p_k}$ is nilpotent. So (2) holds for all $n \geq d_{\mathcal{H}}^2$.

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Putting it all together

$$\text{EEB} \implies \begin{array}{c}\text{eventually}\\ \text{PPT}\end{array} \implies \begin{array}{c}\text{eventually}\\ \bigoplus\text{-irreducible}\end{array} \implies \begin{array}{c}\text{eventually}\\ \bigoplus\text{-primitive}\end{array} \implies \text{EEB}$$

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Let Φ be faithful and EEB. Then

$$\Phi^{nZ} = \bigoplus_j \bigoplus_k (\underbrace{\Phi^Z|_{P_j p_k \mathcal{B}(\mathcal{H}) p_k P_j}}_{\text{primitive}})^n \qquad \forall n \geq d_{\mathcal{H}}^2$$

for $Z =$ least common multiple$(z_1, \ldots, z_N)$, where z_j is the period of $\Phi|_{P_j \mathcal{B}(\mathcal{H}) P_j}$.

Lemma (Rosser and Schoenfeld, 1962)

$\operatorname{argmax}_{n\in\mathbb{N}} \frac{1}{n} \log(\operatorname{lcm}(1, \ldots, n)) = 113$ *and consequently*

$$\operatorname{lcm}(1, \ldots, n) \leq \exp(1.03883n) \qquad \forall n \in \mathbb{N}.$$

Thus, we have the bound

$$Z = \operatorname{lcm}(z_1, z_2, \ldots, z_N) \leq \operatorname{lcm}(1, 2, \ldots, d_{\mathcal{H}}) \leq \exp(1.04 * d_{\mathcal{H}}).$$

Summary

Theorem (H., Rouzé, Stilck França '19)

Let Φ be a faithful quantum channel. Then the following are equivalent:

1. $\{\Phi\}_{n\in\mathbb{N}}$ *is eventually entanglement breaking*
2. $\{\Phi\}_{n\in\mathbb{N}}$ *is eventually PPT*
3. Φ^m *is the direct sum of primitive channels for some* $m \leq d_{\mathcal{H}}^2 \exp(1.04 * d_{\mathcal{H}})$
4. $\Phi^{d_{\mathcal{H}}^2 Z}$ *is the direct sum of primitive channels*

Corollary

Every faithful PPT quantum channel is eventually entanglement breaking.

C.f. Rahaman, Jaques, Paulsen, 2018: unital PPT quantum channels are EEB

Outline

Qualitative level: what evolutions eventually break entanglement?

Quantitative level: when do evolutions eventually break entanglement?

What do we need?

Def. $n_{\mathrm{EB}}(\Phi) = \min\{n : \Phi^n \text{ is EB}\}$.

Note: if $\Phi = \Phi_1 \oplus \Phi_2$ then $n_{\mathrm{EB}}(\Phi) = \max(n_{\mathrm{EB}}(\Phi_1), n_{\mathrm{EB}}(\Phi_2))$.

Recall:

1. if Φ is a faithful quantum channel, then Φ is EEB if and only if $\Phi^{d_{\mathcal{H}}^2 Z}$ is $\bigoplus$-primitive, and $Z \leq \exp(1.04 * d_{\mathcal{H}})$.
2. If Φ is primitive with invariant state σ, then $\Phi^n \otimes \mathrm{id}(\rho_{AB}) \to \sigma \otimes \frac{I}{d_{\mathcal{H}}}$
3. If $\|\rho_{AB} - \sigma \otimes \frac{I}{d_{\mathcal{H}}}\|_2 \leq \frac{\lambda_{\min}(\sigma)}{d}$, then $\rho_{AB} \in \mathrm{SEP}$.

Remains to quantify mixing time

$$n_{\text{mix, 2}}(\Phi) := \min\left\{n : \|\Phi^n \otimes \mathrm{id}(\rho_{AB}) - \sigma \otimes \frac{I}{d_{\mathcal{H}}}\|_2 \leq \frac{\lambda_{\min}(\sigma)}{d} \ \forall \rho_{AB}\right\}$$

for primitive Φ.

Bounds on this are already known!

- E.g. [Temme et al, 2010, Szehr et al 2015]

What do we do?

Quantify convergence for general (non-primitive) maps by comparing to a continuous-time semigroup.

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Why?

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Quantify convergence for general (non-primitive) maps by comparing to a continuous-time semigroup.

Why?

- Provides bounds on the primitive case too
- Might be able to establish bounds on $n_{\mathrm{EB}}(\Phi)$ more directly than via eventually $\bigoplus$-primitive
  - to do: figure out how!
- Might be useful in other contexts
  - quantify convergence for non-EEB channels
  - relate discrete and continuous evolutions

Decoherence-free algebra

Φ: faithful quantum channel. (Dual of) decoherence-free algebra (DFA):

$$\tilde{\mathcal{N}}(\Phi) := \bigoplus_{i=1}^{K} \mathcal{B}(\mathcal{H}_i) \otimes \tau_i.$$

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If $X \in \tilde{\mathcal{N}}(\Phi)$ then $X = \bigoplus_{i=1}^{K} X_i \otimes \tau_i$ and

$$\Phi(X) = \bigoplus_{i=1}^{K} U_i X_{\pi(i)} U_i^{\dagger} \otimes \tau_i \, .$$

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C.f. faithful continuous-time semigroups: $(\Psi_t)_{t\geq 0}$ has an associated space dual to the DFA:

$$\tilde{\mathcal{N}}((\Psi_t)_{t\geq 0}) = \bigoplus_{i=1}^{K'} \mathcal{B}(\mathcal{H}'_i) \otimes \tau'_i .$$

such that if $X \in \tilde{\mathcal{N}}((\Psi_t)_{t\geq 0})$ then

$$\Psi_t(X) = \bigoplus_{i=1}^{K'} \mathrm{e}^{itH_i} X_i \mathrm{e}^{-itH_i} \otimes \tau'_i .$$

Convergence to DFA

Let $E_{\tilde{\mathcal{N}}}$ be the conditional expectation onto $\tilde{\mathcal{N}}((\Psi_t)_{t\geq 0})$. Since $(\Psi_t)_{t\geq 0}$ is faithful, we have *environmental induced decoherence*:

$$\Psi_t(\rho - E_{\tilde{\mathcal{N}}}(\rho)) \xrightarrow{t\to\infty} 0$$

This speed can be quantified by functional inequalities (In the non-primitive case, see: Bardet 2017, Bardet Rouzé 2018, Gao Junge LaRacuente 2018,. . .).

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In discrete time: Let P be the projection onto the peripheral eigenspaces of Φ. Then $\Phi_P = \Phi \circ P$ and

$$\Phi^n(\rho - P(\rho)) \xrightarrow{n\to\infty} 0$$

How to quantify convergence?

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How to quantify convergence? Our approach: relate Φ to a continuous-time semigroup and use continuous-time results.

Step 1: use reference state σ_{tr}

Let

- $\sigma_{\mathrm{tr}} = \frac{1}{d_{\mathcal{H}}} P^*(I)$ (reference state),
- $\Gamma_{\sigma_{\mathrm{tr}}}(X) = \sigma_{\mathrm{tr}}^{1/2} X \sigma_{\mathrm{tr}}^{1/2}$
- $\hat{\Phi} = \Gamma_{\sigma_{\mathrm{tr}}}^{-1} \circ \Phi \circ \Gamma_{\sigma_{\mathrm{tr}}}$, and $\hat{P} = \Gamma_{\sigma_{\mathrm{tr}}}^{-1} \circ P \circ \Gamma_{\sigma_{\mathrm{tr}}}$.

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Why?

1. We can bound

$$\begin{aligned} \|\Phi(\rho) - \Phi \circ P(\rho)\|_2 &\leq \|\Phi(\rho) - \Phi \circ P(\rho)\|_1 \\ &= \|\hat{\Phi}(X) - \hat{\Phi} \circ \hat{P}(X)\|_{1,\sigma_{\mathrm{tr}}} \\ &\leq \|\hat{\Phi}(X) - \hat{\Phi} \circ \hat{P}(X)\|_{2,\sigma_{\mathrm{tr}}} \end{aligned}$$

   where

$$\|X\|_{p,\sigma} := \mathrm{tr}(|\sigma^{\frac{1}{2p}} X \sigma^{\frac{1}{2p}}|^p)^{\frac{1}{p}}$$

2. and $\hat{P}$ is a conditional expectation (whereas P may not be).

Step 2: Relate discrete to continuous

- Let $\mathcal{L}_k^* = (\Phi^*)^k \circ (\hat{\Phi})^k - \mathrm{id}$ for some k.
- Then $\mathcal{L}_k$ is the generator of a continuous-time quantum Markov semigroup $(\Phi_t^{(k)})_{t \geq 0}$.
- If k is larger than the largest Jordan block of Φ and proportional to the order of π, then
  1. DFAs coincide: $\mathcal{N}(\hat{\Phi}) = \mathcal{N}((\Phi_t^{(k)})^*)_{t \geq 0}$
  2. Convergence bound: $\|\hat{\Phi}^k(X) - \hat{\Phi}^k \circ \hat{P}(X)\|_{2,\sigma_{\mathrm{tr}}}^2 \leq (1 - \lambda(\mathcal{L}_k^*))\|X - \hat{P}(X)\|_{2,\sigma_{\mathrm{tr}}}^2$ where
  $$\lambda(\mathcal{L}^*) := \inf_{X \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})} \frac{-\langle X, \mathcal{L}^*(X) \rangle_{\sigma_{\mathrm{tr}}}}{\|X - E_{\mathcal{N}}^*(X)\|_{2,\sigma_{\mathrm{tr}}}^2},$$
  is the Poincaré constant (spectral gap).
  3. $\|\Phi^{nk}(\rho) - \Phi^{nk} \circ P(\rho)\|_1 \leq \frac{1}{\sqrt{\lambda_{\min}(\sigma_{\mathrm{tr}})}}(1 - \lambda(\mathcal{L}_k^*))^{\frac{n}{2}}$

Outlook

- Characterized EEB faithful quantum channels
  - Can remove trace-preserving assumption (ongoing work)
- $\mathrm{PPT}^{\text{finite}}$ ✓
- Obtain a few other results using similar techniques:
  - Lower bounds on EB times (by estimating time-until-PPT)
  - Estimate locally entanglement-annihilating times for product channels (e.g. when is $\Phi^n \otimes \Psi^n(\rho_{AB}) \in \mathrm{SEP}$ for all ρ_{AB})
  - For primitive, unital Φ, estimate time to mixed unitary, i.e. for what n is $\Phi^n(\rho) = \sum_i p_i U_i \rho U_i^\dagger$

Open questions:

- Can one remove the faithful assumption?
- Universal finite bound: PPT^n for some n
  - (i.e. prove $\exists n$ such that for all Φ PPT, Φ^n is EB)
  - Ideally $n = 2$ to prove PPT^2!
- Relax Markovianity?