A universal construction of tight continuity bounds for entropies

arXiv:1706.02212, arXiv:1707.04249, in progress...

Eric P. Hanson, Nilanjana Datta

Univ. Cambridge, DAMTP

UNIVERSITY OF CAMBRIDGE

CCIMI — CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Reducing to the classical case

$\mathcal{H}$: Hilbert space; $\dim \mathcal{H} = d < \infty$; $\mathcal{D} \equiv \mathcal{D}(\mathcal{H})$: set of density matrices;

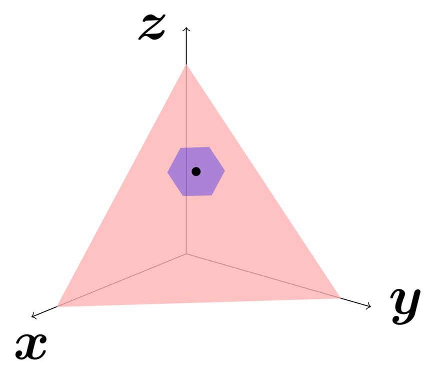
$$B_\varepsilon(\sigma) := \{\rho \in \mathcal{D} : \tfrac{1}{2}\|\rho - \sigma\|_1 \le \varepsilon\},$$

the ε-ball. Work in the basis where $\sigma = \operatorname{Eig}^\downarrow(\sigma)$, where $\operatorname{Eig}^\downarrow(\sigma)$ is the diagonal matrix of eigenvalues of σ in decreasing order. Then for $\rho \in B_\varepsilon(\sigma)$,

$$\frac{1}{2}\sum_{j=1}^d |\lambda_j^\downarrow(\rho) - \lambda_j^\downarrow(\sigma)| = \frac{1}{2}\|\operatorname{Eig}^\downarrow(\rho) - \sigma\|_1 \le \frac{1}{2}\|\rho - \sigma\|_1 \le \varepsilon.$$

As $\operatorname{Eig}^\downarrow(\rho)$ is unitarily equivalent to ρ, for single-partite entropies only need the ℓ^1 ball $B_\varepsilon(q) = \{p \in \operatorname{Simp} : \frac{1}{2}\sum_j |p_j - q_j| \le \varepsilon\}$, for $q = \vec{\lambda}^\downarrow(\sigma)$.

Right: q is the black dot, $B_\varepsilon(q)$ the purple hexagon, in dimensions $d = 3$.



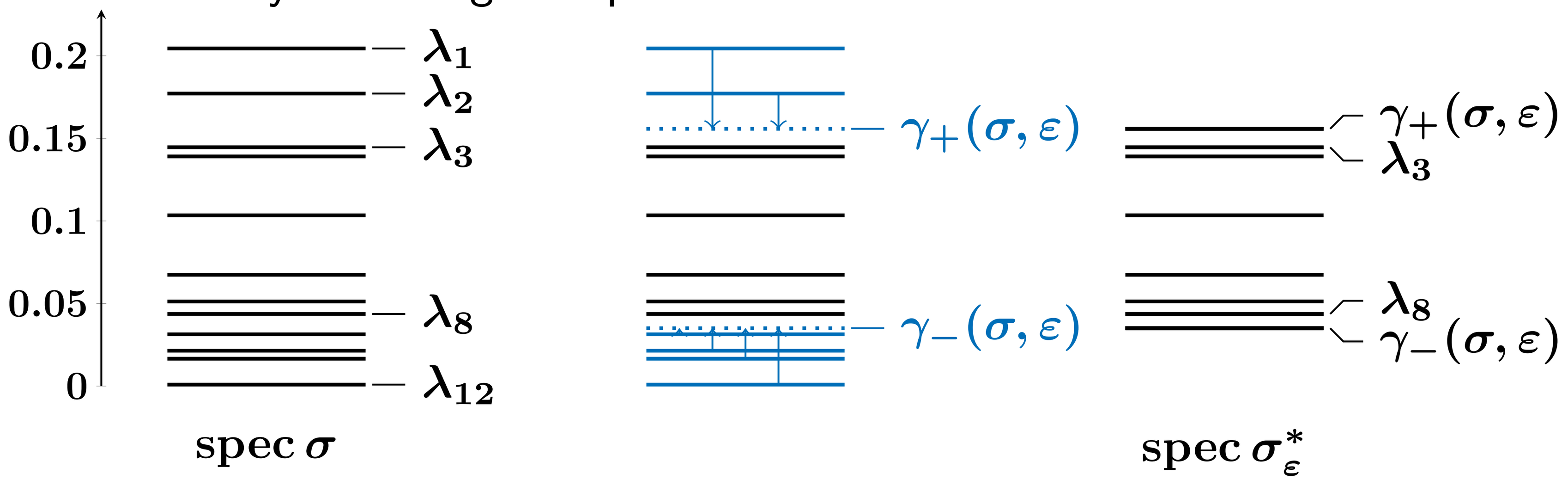
$\lambda_j^\downarrow(\rho)$ denotes the jth largest eigenvalue of ρ.
$\operatorname{Simp} := \{p \in \mathbb{R}^d : p_i \ge 0, \sum_i p_i = 1\}$.

Local extrema in the majorization order

Given $\sigma \in \mathcal{D}$ and $\varepsilon \in (0,1]$, we construct $\sigma_{*,\varepsilon}, \sigma_\varepsilon^* \in B_\varepsilon(\sigma)$ such that

$$\forall\, \omega \in B_\varepsilon(\sigma), \qquad \sigma_\varepsilon^* \prec \omega \prec \sigma_{*,\varepsilon} \tag{1}$$

where $\prec$ is the majorization order (defined below). The state σ_ε^* is constructed by flattening the spectrum:



Specific case: pure state $\psi = \operatorname{diag}(1, 0, \dots, 0)$; in the same basis we have

$$\psi_\varepsilon^* = \begin{cases} \operatorname{diag}(1-\varepsilon, \frac{\varepsilon}{d-1}, \dots, \frac{\varepsilon}{d-1}) & \varepsilon < 1 - \frac{1}{d} \\ \frac{1}{d}\mathbb{1} & \varepsilon \ge 1 - \frac{1}{d}. \end{cases} \tag{2}$$

The order (1) yields *local continuity bound* for H Schur concave: $\forall\, \omega \in B_\varepsilon(\sigma)$,

$$|H(\omega) - H(\sigma)| \le \max\{H(\sigma_\varepsilon^*) - H(\sigma), H(\sigma) - H(\sigma_{*,\varepsilon})\}. \tag{3}$$

Uniform bounds from local majorization

For H Schur concave, $\rho, \sigma \in \mathcal{D}$ with $\frac{1}{2}\|\rho - \sigma\|_1 \le \varepsilon$,

$$\begin{aligned} |H(\rho) - H(\sigma)| &\le \max\{H(\sigma_\varepsilon^*) - H(\sigma), H(\rho_\varepsilon^*) - H(\rho)\} \\ &\le \sup_{\omega \in \mathcal{D}} \Delta_\varepsilon(\omega) \end{aligned} \tag{4}$$

where $\Delta_\varepsilon : \mathcal{D} \to \mathbb{R}_{\ge 0}$ defined by $\Delta_\varepsilon(\omega) := H(\omega_\varepsilon^*) - H(\omega)$.

Goal: maximize Δ_ε over $\mathcal{D}$ for some useful choices of H. (E.g. $H(\rho) = S(\rho) := -\operatorname{Tr}[\rho\log\rho]$ the von Neumann entropy.)

Definition of majorization

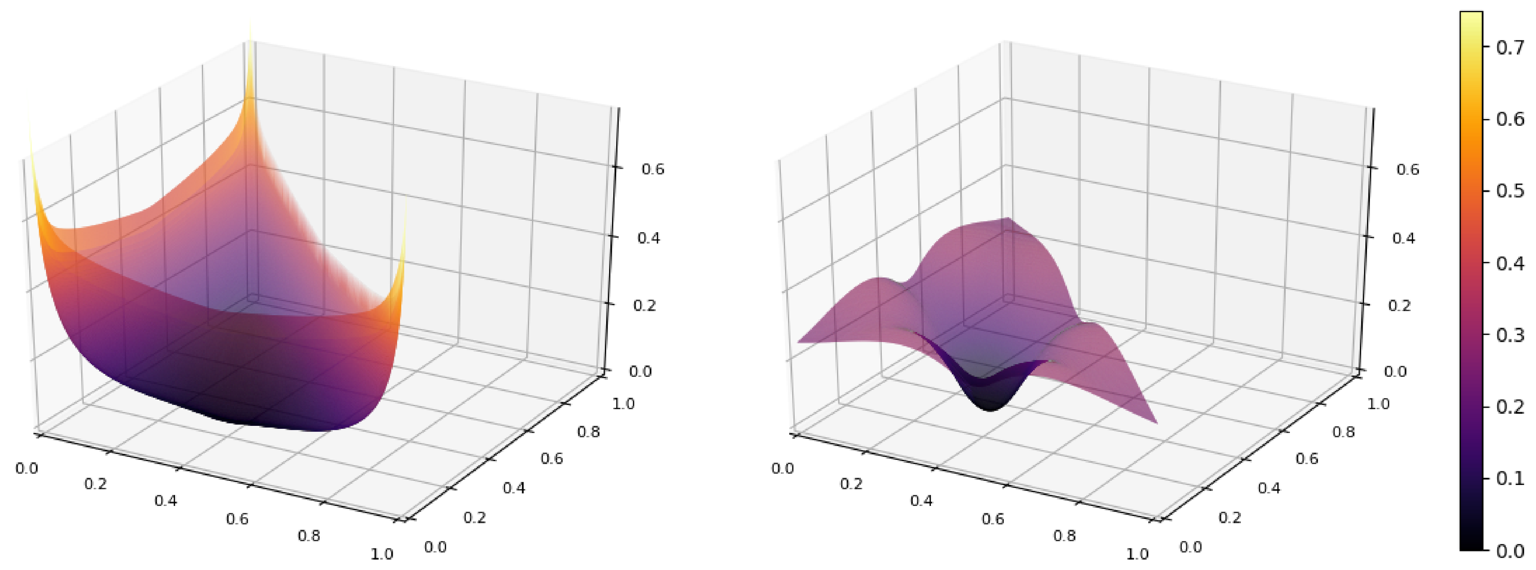
- We say $\rho \prec \sigma$ if the vector of eigenvalues $\vec{\lambda}^\downarrow(\rho) \prec \vec{\lambda}^\downarrow(\sigma)$.
- For $p \in \mathbb{R}^d$, set $p^\downarrow$ as the permutation of p in decreasing order. Then for $p, q \in \mathbb{R}^d$, we say $p \prec q$ if $\sum_{i=1}^k p^\downarrow(i) \le \sum_{i=1}^k q^\downarrow(i)$ for each $k = 1, 2, \dots, d$, with equality for $k = d$.
- $H : \mathcal{D} \to \mathbb{R}$ is *Schur concave* if $\rho \prec \sigma \implies H(\rho) \ge H(\sigma)$.

Right: Comparison of continuity bounds for the Rényi entropy with $\alpha = \frac{1}{2}$ and $d = 5$. The small crosses are points corresponding to the right-hand side of (3) for 500 randomly chosen states σ.

[Ras11]: Rastegin. arXiv: 1012.5356. [Che+17]: Chen et al. 1701.02398.

Plotting Δ_ε for α-Rényi entropies

Take $H(\rho) = S_\alpha(\rho) := \frac{1}{1-\alpha}\log[\operatorname{Tr}(\rho^\alpha)]$, the α-Rényi entropy. In $d = 3$, we parametrize $\sigma = \operatorname{diag}(x, y, 1-x-y)$ and plot $(x,y) \mapsto \Delta_\varepsilon(\sigma)$. Left: $\alpha = 0.5$, right: $\alpha = 2.5$.



Observation: for $\alpha \le 1$, the maxima occur at $(0,0)$, $(1,0)$, and $(0,1)$ i.e. when σ is a pure state. For $\alpha > 1$, the maxima change with α.

Schur convexity of Δ_ε

Δ_ε is not convex even for the von Neumann entropy (counterexamples). However, **Δ_ε is Schur convex**, when $H = H_{(h,\phi)}$ for

$$H_{(h,\phi)}(\rho) := h(\operatorname{Tr}[\phi(\rho)]) \tag{5}$$

where $h : \mathbb{R} \to \mathbb{R}$ is concave, $\phi : [0,1] \to \mathbb{R}$ strictly concave, $\phi(0) = 0$, and $h(\phi(1)) = 0$. Examples include the von Neumann entropy, the Tsallis entropy, and the α-Rényi entropy for $\alpha \in (0,1)$.

Main results: Tight uniform bounds and Lipschitz continuity (LC)

1. Schur convex on $\mathcal{D}$ implies maxima are pure. By (4), if $\frac{1}{2}\|\rho - \sigma\|_1 \le \varepsilon$,

$$|H_{(h,\phi)}(\rho) - H_{(h,\phi)}(\sigma)| \le H_{(h,\phi)}(\psi_\varepsilon^*) \tag{6}$$

for any pure state ψ, where ψ_ε^* is given by (2). In particular, for $\alpha \in (0,1)$,

$$|S_\alpha(\rho) - S_\alpha(\sigma)| \le \tfrac{1}{1-\alpha}\log((1-\varepsilon)^\alpha + (d-1)^{1-\alpha}\varepsilon^\alpha) \tag{7}$$

for $\varepsilon < 1 - \frac{1}{d}$, and the trivial bound $\log d$ for $\varepsilon \ge 1 - \frac{1}{d}$. Moreover, equality in (6) or (7) occurs if and only if one of the two states (say, σ) is pure, and $\rho = \sigma_\varepsilon^*$ given by (2).

2. S_α is LC on $\mathcal{D}$ if and only if $\alpha > 1$. That is, $\exists C_\alpha < \infty$ such that

$$|S_\alpha(\rho) - S_\alpha(\sigma)| \le C_{\alpha,d}\|\rho - \sigma\|_1 \qquad \forall \rho, \sigma \in \mathcal{D}. \tag{8}$$

In fact, $C_{\alpha,d} = \frac{\alpha}{\alpha-1}\frac{1}{2\ln(2)}c_{\alpha,d}$ where $c_{2,d} \sim \sqrt{d}$, $c_{\alpha,d} \le d$.

Proof sketch of (6) $\mathcal{M}_\varepsilon(\omega) := \omega_\varepsilon^*$. Semigroup: $\mathcal{M}_{\varepsilon_1+\varepsilon_2}(\omega) = \mathcal{M}_{\varepsilon_1}(\mathcal{M}_{\varepsilon_2}(\omega))$. Then $\Delta_{\varepsilon_1+\varepsilon_2}(\omega) = \Delta_{\varepsilon_1}(\mathcal{M}_{\varepsilon_2}(\omega)) + \Delta_{\varepsilon_2}(\omega)$. By iterating, only need to prove Schur convexity of Δ_ε for small ε, where $\mathcal{M}_\varepsilon(\sigma)$ only moves the smallest and largest eigenvalues of σ. Reduces to concavity of ϕ on $\mathbb{R}$.

Performance of the bounds by example