Entanglement-breaking properties of repeated interaction systems

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Overview

What are repeated interaction systems? What did we prove?

Outline

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Framework I: Finite-dimensional quantum systems

Definition (Quantum system)

A *quantum system* $\mathcal{S}$ is associated to a Hilbert space $\mathcal{H}_\mathcal{S}$ (finite dimensions: $\mathcal{H}_\mathcal{S} \cong \mathbb{C}^{d_\mathcal{S}}$ for some dimension $d_\mathcal{S}$)

Definition (State)

A *state* (aka *density matrix*) on $\mathcal{S}$ is a $d_\mathcal{S} \times d_\mathcal{S}$ Hermitian matrix ρ such that $\mathrm{tr}[\rho] = 1$ and $\rho \geq 0$ (i.e. positive semi-definite: all eigenvalues are ≥ 0).

Definition (Combining systems)

A composite system made up of two systems $\mathcal{S}$ and $\mathcal{E}$ is given by $\mathcal{S}\mathcal{E}$ with Hilbert space $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$.

Framework II: Quantum channels

Definition (Quantum channel)

A *quantum channel* $\mathcal{N}$ on $\mathcal{S}$ is a linear map which takes states on $\mathcal{S}$ to states on $\mathcal{S}$ such that $\mathcal{N} \otimes \mathrm{id}_{\mathcal{E}}$ takes states on $\mathcal{S}\mathcal{E}$ to states on $\mathcal{S}\mathcal{E}$, for any finite-dimensional system $\mathcal{E}$.

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A *quantum channel* $\mathcal{N}$ on $\mathcal{S}$ is a linear map which takes states on $\mathcal{S}$ to states on $\mathcal{S}$ such that $\mathcal{N} \otimes \mathrm{id}_{\mathcal{E}}$ takes states on $\mathcal{SE}$ to states on $\mathcal{SE}$, for any finite-dimensional system $\mathcal{E}$.

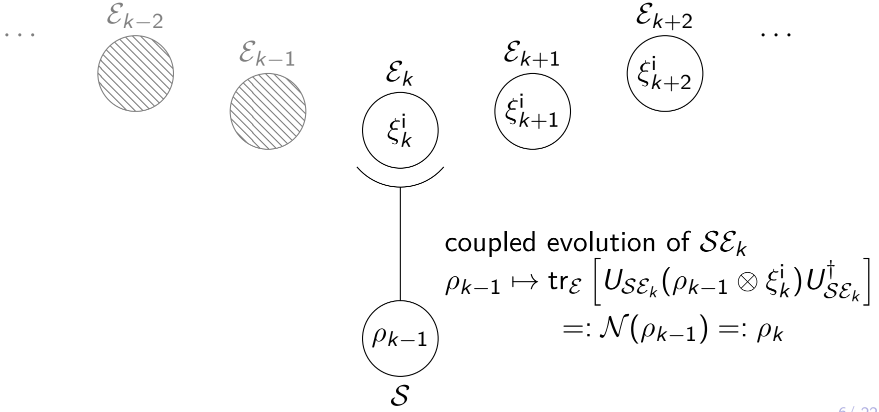
- Mathematically, this corresponds to to $\mathcal{N}$ being completely-positive and trace-preserving (CPTP).
- General form of open time evolution; e.g.

$$\mathcal{N}(\rho) = \mathrm{tr}_{\mathcal{E}} \left[U_{\mathcal{SE}} (\rho \otimes \xi_{\mathcal{E}}) U_{\mathcal{SE}}^{\dagger} \right]$$

  is CPTP for any unitary $U_{\mathcal{SE}}$ on $\mathcal{SE}$ and state $\xi_{\mathcal{E}}$ on $\mathcal{E}$.

Repeated interaction systems (RIS)

- System $\mathcal{S}$ in initial state ρ_0^{i}
- Interacts with probe $\mathcal{E}_1$ which is in initial state ξ_1^{i} via a joint evolution $U_{\mathcal{S}\mathcal{E}_1}$.
- Trace out the probe, yielding state ρ_1 on $\mathcal{S}$.
- Repeat! $\rho_k = \mathcal{N}^k(\rho_0)$ if all probes and couplings identical.



Example RIS: Haroche experiment (doi:10.1038/nature05589)

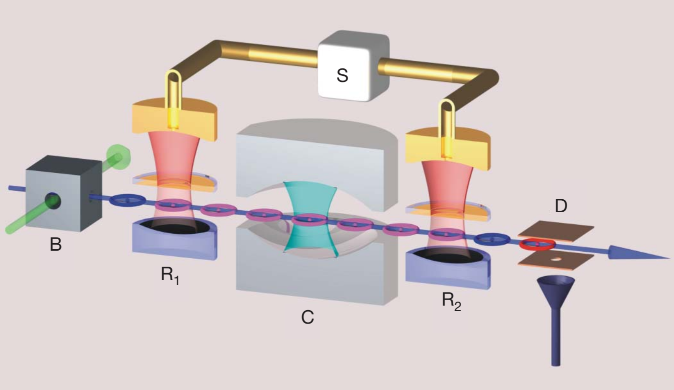


Figure 1 | Experimental set-up. Samples of circular Rydberg atoms are prepared in the circular state *g* in box B, out of a thermal beam of rubidium atoms, velocity-selected by laser optical pumping. The atoms cross the cavity C sandwiched between the Ramsey cavities R_1 and R_2 fed by the classical microwave source S, before being detected in the state selective field ionization detector D. The R_1–C–R_2 interferometric arrangement, represented here cut by a vertical plane containing the atomic beam, is enclosed in a box at 0.8 K (not shown) that shields it from thermal radiation and static magnetic fields.

Abstraction: from RIS to $\mathcal{N}^k$

In the following, we will just consider the quantum channel

$$\mathcal{N}(\rho) := \mathrm{tr}_{\mathcal{E}}\left[U_{\mathcal{S}\mathcal{E}}(\rho \otimes \xi^{\mathrm{i}})U_{\mathcal{S}\mathcal{E}}^{\dagger}\right]$$

in the case in which all the probes are identical. $\mathcal{N}$ is called the *reduced dynamics* of the repeated interaction system, and governs the time evolution at each step, instead of ξ^{i}, $U_{\mathcal{S}\mathcal{E}}$, etc.

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We also study $\mathcal{N}^k := \underbrace{\mathcal{N} \circ \mathcal{N} \circ \cdots \circ \mathcal{N}}_{k \text{ times}}$ for natural numbers k, which gives the time evolution from the initial state to the state after step k.

Entanglement

Definition (Entanglement)

Consider a state $\rho_{\mathcal{SR}}$ on $\mathcal{SR}$, where $\mathcal{R}$ is just another system, called the reference. We say $\rho_{\mathcal{SR}}$ is *entangled* if

$$\rho_{\mathcal{SR}} \neq \sum_i p_i \sigma_{\mathcal{S}}^{(i)} \otimes \tau_{\mathcal{R}}^{(i)}$$

for any probabilities $\{p_i\}_i$, states $\{\sigma_{\mathcal{S}}^{(i)}\}_i$ on $\mathcal{S}$, and states $\{\tau_{\mathcal{R}}^{(i)}\}_i$ on $\mathcal{R}$. In other words, $\rho_{\mathcal{SR}}$ is entangled if it cannot be written as a (weighted) average of product states.

Entanglement-breaking channels

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A quantum channel $\mathcal{N}$ is *entanglement-breaking* (EB) if the output state $\mathcal{N} \otimes \mathrm{id}_{\mathcal{R}}(\rho_{\mathcal{SR}})$ is not entangled for any input state $\rho_{\mathcal{SR}}$.

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There are several characterizations of entanglement-breaking channels; while computationally it is NP-hard to check in general, it is fairly well understood mathematically. We will use the following characterization: $\mathcal{N}$ is *entanglement-breaking* iff $\mathcal{N} \otimes \mathrm{id}_{\mathcal{R}}(\Omega)$ is non-entangled, where

$$\Omega := |\Omega\rangle\langle\Omega|; \quad |\Omega\rangle := \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle \tag{1}$$

where $\mathcal{R}$ is such that $d_{\mathcal{R}} = d_{\mathcal{S}} \equiv d$. Ω is called the *maximally entangled state*.

Instead of asking if $\mathcal{N}$ is EB, we ask:

Main question

What quantum channels $\mathcal{N}$ are *eventually entanglement-breaking* (EEB)? In other words, for which channels $\mathcal{N}$ is it true that for some number k, the channel $\mathcal{N}^k$ is entanglement-breaking?

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- Could check each k: is $\mathcal{N}$ EB? What about $\mathcal{N}^2$? $\mathcal{N}^3$? ...
- Instead: can we characterize EEB channels just looking at $\mathcal{N}$ itself? (almost)

Motivation: connection with long-time convergence

We were led to this question by the following observation:

- Consider a quantum channel $\mathcal{N}$ on $\mathcal{S}$ such that there exists a full-rank state σ on $\mathcal{S}$ and we have the long-time convergence

$$\mathcal{N}^n(\rho) \xrightarrow{n\to\infty} \sigma \qquad \forall \rho \text{ on } \mathcal{S} \tag{$\star$}$$

- Then $\mathcal{N}^n$ is entanglement-breaking for some finite $n \in \mathbb{N}$.

(Relevance: $(\star)$ is very common! $\mathcal{N}$ is called *primitive* if $(\star)$ holds for some full-rank state σ.)

Proof.

1. First, it suffices to check if $\mathcal{N} \otimes \mathrm{id}_{\mathcal{R}}(\Omega)$ is entangled.
2. Fact: If $\mathcal{N}^n(\rho) \to \sigma$ for all states ρ, then $\mathcal{N}^n(X) \to \mathrm{tr}[X]\sigma$ for all matrices X.
3. Then,

$$\begin{aligned}
\mathcal{N}^n \otimes \mathrm{id}_{\mathcal{R}}(\Omega_{\mathcal{SR}}) &= \frac{1}{d}\sum_{ij} \mathcal{N}^n(|i\rangle\langle j|_{\mathcal{S}}) \otimes |i\rangle\langle j|_{\mathcal{R}} \\
&\to \frac{1}{d}\sum_{ij} \mathrm{tr}[|i\rangle\langle j|_{\mathcal{S}}]\,\sigma \otimes |i\rangle\langle j|_{\mathcal{R}} \\
&= \sigma \otimes \frac{1}{d}\sum_{ij} \delta_{ij}|i\rangle\langle j|_{\mathcal{R}} \\
&= \sigma \otimes \frac{1}{d}\sum_{i} |i\rangle\langle i|_{\mathcal{R}} = \sigma \otimes \tau
\end{aligned}$$

where $\tau = \frac{1}{d_{\mathcal{R}}} I_{\mathcal{R}}$. A product state!

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Corollary (H., Rouzé, Stilck França '19; Gurvits & Barnum '02)

A state $\omega_{\mathcal{SR}}$ *is non-entangled whenever* $\|\omega_{\mathcal{SR}} - \sigma \otimes \tau\|_2 \leq \lambda_{min}(\sigma)\lambda_{min}(\tau)$ *for some states* σ *and* τ*. Here,* $\|X\|_2 := \sqrt{\mathrm{tr}[X^\dagger X]}$ *and* $\lambda_{min}(X)$ *is the smallest eigenvalue of* X*.*

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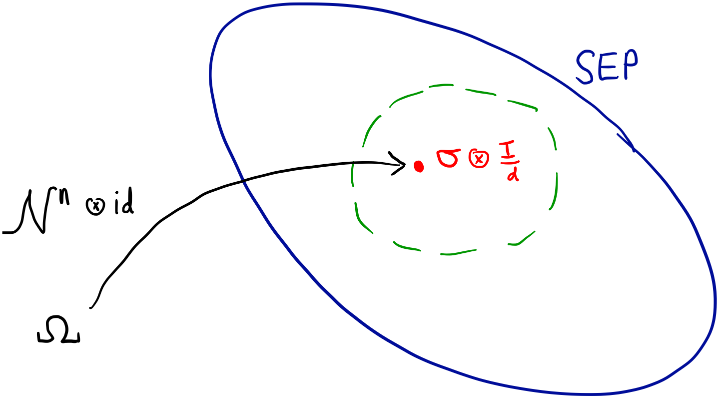
This is enough to finish the proof:

4. Since $\mathcal{N}^n \otimes \mathrm{id}_{\mathcal{R}}(\Omega_{\mathcal{SR}}) \xrightarrow{n\to\infty} \sigma \otimes \tau$, for n large enough, it holds that

$$\|\mathcal{N}^n(\Omega_{\mathcal{SR}}) - \sigma \otimes \tau\|_2 \leq \lambda_{\min}(\sigma)\lambda_{\min}(\tau) = \frac{1}{d}\lambda_{\min}(\sigma).$$

Summary of the proof



where SEP is the set of unentangled states (i.e. *separable* states).

The question

We've seen that any quantum channel $\mathcal{N}$ with the long-time convergence behavior

$$\mathcal{N}^n(\rho) \xrightarrow{n\to\infty} \sigma \qquad \forall\rho \text{ on } \mathcal{S} \tag{$\star$}$$

is entanglement-breaking in finite time (i.e. *eventually entanglement-breaking*, EEB). Intuition: it forgets the initial state, and hence initially-present correlations too.

So what happens if $\mathcal{N}$ is not primitive? I.e., $\mathcal{N}$ does not satisfy $(\star)$?

What happens if $\mathcal{N}$ is not primitive? (I)

Recall: primitive means for some full-rank σ,

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No: Consider if $\mathcal{H}_\mathcal{S} = \mathcal{H}_\mathcal{S}^1 \oplus \mathcal{H}_\mathcal{S}^2$, and $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$. That is, for a state ρ on $\mathcal{S}$,

$$\mathcal{N}\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\right) := \begin{pmatrix} \mathcal{N}_1(\rho_{11}) & 0 \\ 0 & \mathcal{N}_2(\rho_{22}) \end{pmatrix}$$

where each ρ_{ij} is a block-matrix, namely the part of ρ mapping $\mathcal{H}_\mathcal{S}^j \to \mathcal{H}_\mathcal{S}^i$. Then, if both $\mathcal{N}_1$ satisfies ($\star$) with some σ_1 and $\mathcal{N}_2$ satisfies ($\star$) with some state σ_2, then similarly as before, $\mathcal{N}$ is EEB.

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But,

$$\mathcal{N}^n(\rho) \xrightarrow{n\to\infty} \mathrm{tr}[\rho_{11}]\sigma_1 \oplus \mathrm{tr}[\rho_{22}]\sigma_2,$$

so $\mathcal{N}$ does not satisfy ($\star$) (final state still depends on the input).

What happens if $\mathcal{N}$ is not primitive? (II)

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No: consider if for some k, $\mathcal{N}^k$ is the sum of primitive channels, but $\mathcal{N}$ itself is not. E.g.

$$\mathcal{N}\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\right) := \begin{pmatrix} \mathrm{tr}(\rho_{22})\sigma_1 & 0 \\ 0 & \mathrm{tr}(\rho_{11})\sigma_2 \end{pmatrix}$$

for some full-rank states σ_1, σ_2. Then

$$\mathcal{N}^2\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\right) = \begin{pmatrix} \mathrm{tr}(\rho_{11})\sigma_1 & 0 \\ 0 & \mathrm{tr}(\rho_{22})\sigma_2 \end{pmatrix}$$

is the direct sum of primitive channels. (Note: $\mathcal{N}^3$ is again not a direct sum of primitive channels. But still, $(\mathcal{N}^2)^k$ is EB for some k, and hence $\mathcal{N}^{2k}$ is EB, so $\mathcal{N}$ is still EEB).

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Note: this assumption is much weaker than primitivity; we do not require $\mathcal{N}^n(\rho) \xrightarrow{n\to\infty} \sigma$, and σ may be non-unique.

A channel $\mathcal{N}$ being *not* faithful is equivalent to saying there exists an orthogonal projection π such that for every state, $\mathrm{tr}[\pi\mathcal{N}^n(\rho)] \xrightarrow{n\to\infty} 0$.

So a faithful channel is one which does not eventually move every state out of some subspace, i.e. there is no transient subspace.

When is a channel EEB?

In fact, we can say a bit more:

Theorem (H., Rouzé, Stilck França '19)

Let $\mathcal{N}$ be a faithful quantum channel. Then the following are equivalent:

1. *$\mathcal{N}$ is eventually entanglement breaking*
2. *$\mathcal{N}^m$ is the direct sum of primitive channels for some $m \leq d_{\mathcal{H}}^2 \exp(1.04 * d_{\mathcal{H}})$*
3. *$\mathcal{N}^{d^2 Z}$ is the direct sum of primitive channels, where Z is the least common multiple of the periods of the irreducible components of $\mathcal{N}$.*

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Note: we aren't sure what happens in the non-faithful case.

Irreducible components?

$\mathcal{N}$ is *irreducible* if only trivial blocks preserved, i.e.

$$\mathcal{N}(P\mathcal{B}(\mathcal{H})P) \subseteq P\mathcal{B}(\mathcal{H})P \implies P \in \{0, I\}$$

for orthogonal projections P. Here, $\mathcal{B}(\mathcal{H})$ is the set of matrices on $\mathcal{H}$.

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Any faithful channel can be decomposed into irreducible components, in the following sense:

Lemma (Carbone & Pautrat 2016)

Let $\mathcal{N}$ be a faithful quantum channel. Then there exist orthogonal projections $P_1, P_2, \ldots, P_N$ such that

1. $\sum_j P_j = I$,
2. *Block-preserving:* $\mathcal{N}(P_j\mathcal{B}(\mathcal{H})P_k) \subseteq P_j\mathcal{N}(\mathcal{B}(\mathcal{H}))P_k \qquad \forall j, k$
3. $\mathcal{N}|_{P_j\mathcal{B}(\mathcal{H})P_j}$ *is irreducible*

Period?

An irreducible channel has a *period*. It turns out, given an irreducible channel $\mathcal{N}$ there exists an orthogonal family of projections $\pi_1, \ldots, \pi_z$ for some number z such that

$$\mathcal{N}(\pi_i X \pi_j) = \pi_{i-1} \mathcal{N}(X) \pi_{j-1}.$$

The number z is called the *period*. Note the nice property that by repeatedly applying the above,

$$\mathcal{N}^z(\pi_i X \pi_j) = \pi_i \mathcal{N}^z(X) \pi_j.$$

That is, $\mathcal{N}^z$ preserves these blocks.