

Part III, Lent 2018
Quantum Information Theory

Example Sheet 2

Exercise 1

1. Prove that if A is positive semi-definite, then A is Hermitian.
2. The action of a flip operator $\mathbb{F} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\mathcal{H}_A, \mathcal{H}_B \simeq \mathbf{C}^d$, is defined through the following equation:

$$\mathbb{F}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle,$$

where $\{|i\rangle\}_{i=1}^d$ denotes an orthonormal basis of \mathbf{C}^d . Write an explicit expression for \mathbb{F} in terms of the basis vectors $|i\rangle$. What are the eigenvalues of \mathbb{F} and what are their multiplicities? Express \mathbb{F} in terms of the MES

$$|\Omega\rangle := \frac{1}{\sqrt{d}}|i\rangle \otimes |i\rangle.$$

Hint: Think of what mathematical operation you would use to relate one to the other.

Use the relation that you obtain to rewrite equations (1.4) and (1.5) of Lemma 1 of Notes 7 (on the course webpage) using the flip operator instead of the MES.

Exercise 2 Show that any density operator of a qubit can be written as

$$\varrho = \frac{1}{2}(\mathbf{1} + \vec{s} \cdot \vec{\sigma}) = \frac{1}{2} \sum_{k=0}^3 s_k \sigma_k, \quad (1)$$

where $\vec{s} = (s_1, s_2, s_3)$ is a real three-dimensional vector such that $\|\vec{s}\| \leq 1$. In the above $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_1, \sigma_2, \sigma_3$ being the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. Moreover, σ_0 denotes the 2×2 identity matrix.

The vector \vec{s} is usually referred to as the Bloch vector. The set of all Bloch vectors defines a sphere which is called the *Bloch sphere*. It provides a useful geometrical representation of the states of a qubit.

Show that $s_k = \text{Tr} \sigma_k \varrho$. Show that $\|\vec{s}\| = 1$ for a pure state ρ . What is the Bloch representation of the completely mixed state $\rho = \mathbf{I}/2$? What do the North Pole and the South Pole of the Bloch sphere correspond to?

Exercise 3 Given the density matrix

$$\rho = \begin{pmatrix} \frac{3}{5} & \frac{1}{4} - i\frac{1}{6} \\ \frac{1}{4} + i\frac{1}{6} & \frac{2}{5} \end{pmatrix}$$

what is the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ for ρ ? Is it a pure state or a mixed state? What is the probability that a measurement of the spin of the qubit along the Z -axis will yield a value $+1$?

Exercise 4 Show that the set of density operators acting on a Hilbert space \mathcal{H} , where $\dim \mathcal{H} = d$ is a *convex subset* of the real vector space of $d \times d$ Hermitian matrices. Show that pure states are extremal points of this set.

Exercise 5 Find the Schmidt ranks for each of the following states

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |44\rangle) & |\phi_2\rangle &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\ |\phi_3\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) & |\phi_4\rangle &= \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle) \end{aligned}$$

Exercise 6 Let $|\Psi_{AR}\rangle$ and $|\Phi_{AR}\rangle$ be two purifications of a state ρ of a system A to a composite system AR . Prove that there exists a unitary transformation U_R acting on system R alone such that

$$|\Phi_{AR}\rangle = (\mathbf{I}_A \otimes U_R)|\Psi_{AR}\rangle.$$

Exercise 7 (Unitary freedom in the Kraus decomposition)

Suppose $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are Kraus operators giving rise to quantum operations Λ and Λ' , respectively. Prove that $\Lambda = \Lambda'$ if and only if there exist complex numbers u_{ij} such that $A_i = \sum_{j=1}^n u_{ij} B_j$, and u_{ij} are the elements of an $n \times n$ unitary matrix.

Exercise 8 (Number of Kraus operators)

1. Consider a quantum operation $\Lambda : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$. What is the maximal rank of its Choi state $J(\Lambda)$?
2. Prove that any quantum operation Λ on the state ρ of a system with a d -dimensional Hilbert space, there exists a Kraus decomposition with at most d^2 elements, i.e.,

$$\Lambda(\rho) = \sum_{i=1}^n A_i \rho A_i^\dagger,$$

where $n \leq d^2$.

Exercise 9 Consider two quantum operations Λ_1 and Λ_2 acting on a single qubit, having Kraus representations

$$\Lambda_1(\rho) = \sum_{k=1}^2 A_k \rho A_k^\dagger; \quad \Lambda_2(\rho) = \sum_{k=1}^2 V_k \rho V_k^\dagger,$$

with

$$A_1 = \frac{1}{\sqrt{2}}\sigma_0; \quad A_2 = \frac{1}{\sqrt{2}}\sigma_z$$

and

$$V_1 = |0\rangle\langle 0|; \quad V_2 = |1\rangle\langle 1|.$$

How do the actions of Λ_1 and Λ_2 differ from each other?

Exercise 10 Show that the following three operators form a POVM.

$$\begin{aligned} E_1 &= \frac{\sqrt{2}}{1 + \sqrt{2}} |1\rangle\langle 1| \\ E_2 &= \frac{\sqrt{2}}{2 + 2\sqrt{2}} (|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \\ E_3 &= \mathbb{1} - E_1 - E_2 \end{aligned}$$

Suppose Alice gives Bob a state prepared in one of the two states $|\psi_1\rangle = |0\rangle$ or $|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Show that if Bob does a measurement characterized by these POVM elements on the state he receives, he never makes an error of misidentification. Discuss the possible outcomes.