Quantum Information Theory, Lent 2018

Example Sheet 3: Quantum Entropies and Distance Measures

Exercise 1. Given two qubit states ρ and ω with Bloch vectors $\vec{r}$ and $\vec{s}$ respectively, show that

$$\|\rho - \omega\|_1 = \|\vec{r} - \vec{s}\|_2. \tag{1}$$

The two norm of a vector $\vec{v} = (v_x, v_y, v_z)$ is $\|\vec{v}\|_2 = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

Exercise 2. One Fuchs and van de Graaf inequality.

1. Show that for two pure *qubit* states ψ and ϕ, we have that

$$D(\psi, \phi) = \sqrt{1 - F(\psi, \phi)^2}. \tag{2}$$

   where D is the trace distance, and F the fidelity, defined in the lectures.

2. Using this, show that for any two mixed qubit states ρ and σ,

$$D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}. \tag{3}$$

Exercise 3. Show that for any two states ρ and σ and any unitary U,

$$F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma) \tag{4}$$

by using the fact that for any positive semi-definite operator A and unitary U, we have $\sqrt{UAU^\dagger} = U\sqrt{A}U^\dagger$.

Exercise 4. Show that the fidelity is jointly concave. That is, given any finite probability distribution $\{p_i\}_{i=1}^n$ and quantum states ρ_i and σ_i for $i = 1, \ldots, n$,

$$F\left(\sum_{i=1}^n p_i\rho_i, \sum_{i=1}^n p_i\sigma_i\right) \geq \sum_{i=1}^n p_i F(\rho_i, \sigma_i). \tag{5}$$

Hint: Choose purifications ϕ_i of ρ_i and ψ_i of σ_i such that $F(\rho_i, \sigma_i) = \langle\phi_i|\psi_i\rangle$, using Uhlmann's theorem.

Exercise 5. Show that the fidelity is *multiplicative under tensor products.* That is,

$$F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = F(\rho_1, \sigma_1)F(\rho_2, \sigma_2). \tag{6}$$

Hint: first show that $\|A \otimes B\|_1 = \|A\|_1\|B\|_1$.

Exercise 6. Let $\rho = \sum_i p_i\rho_i$ where ρ_i are density matrices which have support on orthogonal subspaces. Then prove that

$$S\left(\sum_i p_i\rho_i\right) = H(p) + \sum_i p_i S(\rho_i)$$

where $p = \{p_i\}$ is a probability distribution, and $H(p)$ is the corresponding Shannon entropy.

Exercise 7. Prove that the von Neumann entropy is a concave function of its inputs, i.e., given probabilities $p_i \geq 0$, $\sum_{i=1}^r p_i = 1$, and corresponding density operators ρ_i:

$$S\left(\sum_{i=1}^r p_i\rho_i\right) \geq \sum_{i=1}^r p_i S(\rho_i). \tag{7}$$

Show that if each $p_i > 0$ and $\rho_i \neq \rho_j$ for $i \neq j$, then the inequality (7) is strict. Note: means the von Neumann entropy is *strictly concave.*

[Hint: Consider the state $\sigma_{AB} := \sum_i p_i\rho_i \otimes |i\rangle\langle i|$, its reduced states σ_A, σ_B and use subadditivity.]

Exercise 8. Consider the following *classical-quantum* or cq-states on the Hilbert space $\mathbb{C}^n \otimes \mathcal{H}$:

$$\rho = \sum_{i=1}^n p_i|i\rangle\langle i| \otimes \rho_i \qquad \sigma = \sum_{i=1}^n p_i|i\rangle\langle i| \otimes \sigma_i$$

where $\rho_i, \sigma_i \in \mathcal{D}(\mathcal{H})$ and $\{p_i\}$ is a probability distribution. Evaluate the quantum relative entropy $D(\rho\|\sigma)$ and use the result to prove that

$$D\left(\sum_i p_i\rho_i \middle\| \sum_i p_i\sigma_i\right) \leq \sum_i p_i D(\rho_i\|\sigma_i),$$

i.e. joint convexity of the quantum relative entropy.

Exercise 9. Let $\rho_{AB} = \sum_i p_i \rho^i_{AB}$ be the state of a bipartite quantum system AB. Using the joint convexity of the relative entropy, prove that the quantum conditional entropy is concave in the state ρ_{AB}.

Exercise 10. Let $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_{B'}$ be Hilbert spaces and $\rho \in \mathcal{D}(\mathcal{H}_{AB})$ be a state. Further, let $\Lambda : \mathcal{H}_B \to \mathcal{H}_{B'}$ be a CPTP map and define

$$\sigma_{AB'} = (\mathrm{id}_A \otimes \Lambda)\rho_{AB}.$$

Prove that

$$S(A|B)_{\rho_{AB}} \leq S(A|B')_{\sigma_{AB'}}.$$

[Hint: Use strong subadditivity.]

Exercise 11. Projective measurements do not decrease von Neumann entropy Suppose a projective measurement described by a set of projection operators $\{P_i\}$ is performed on a quantum system, but we never learn the result of the measurement. If the state of the system before the measurement was ρ then the state after the measurement is given by

$$\rho' = \sum_i P_i \rho P_i.$$

Prove that the entropy of this final state is at least as great as the original entropy:

$$S(\rho') \geq S(\rho),$$

with equality if and only if $\rho = \rho'$.