# Quantum Information Theory, Lent 2018

## Example Sheet 4:

**Exercise 1 The other Fuchs and van de Graaf inequality.**

1. Let $\{E_m\}_{m\in M}$ be a POVM, where $M$ is a finite set. Given two states $\rho_A$ and $\sigma_A$, use the Cauchy-Schwarz inequality to show that

$$F(\rho_A, \sigma_A) \leq \sum_{m\in M} \sqrt{\operatorname{tr}[E_m\rho]\operatorname{tr}[E_m\sigma]} = F(\tilde{\rho}_M, \tilde{\sigma}_M) \tag{1}$$

where

$$\tilde{\rho}_M = \sum_{m\in M} \operatorname{tr}[E_m\rho] |m\rangle\langle m|, \qquad \tilde{\sigma}_M = \sum_{m\in M} \operatorname{tr}[E_m\sigma] |m\rangle\langle m| \tag{2}$$

are diagonal states encoding probabilities of the measurement outcomes $m$, in a Hilbert space $\mathcal{H}_M$ of dimension $M$ with orthonormal basis $\{|m\rangle\}_{m\in M}$.

*Hint: use the polar decomposition* $\sqrt{\rho}\sqrt{\sigma} = |\sqrt{\rho}\sqrt{\sigma}|U$.

2. Let $\rho$ and $\sigma$ be mixed states and assume $\rho$ is invertible.

Show that if $\hat{M} := \rho^{-1/2}|\rho^{1/2}\sigma^{1/2}|\rho^{-1/2}$ has spectral decomposition $\hat{M} = \sum_m \lambda_m |m\rangle\langle m|$, then

$$|m\rangle\langle m| \sqrt{\rho} = \frac{1}{\lambda_m} |m\rangle\langle m| \sigma^{1/2} U^\dagger, \tag{3}$$

where $U$ is unitary and defined by the polar decomposition $\sqrt{\rho}\sqrt{\sigma} = |\sqrt{\rho}\sqrt{\sigma}|U$.

*Hint: start by showing that* $|m\rangle\langle m| M = \lambda_m |m\rangle\langle m|$.

3. Show that projective measurement $\{E_m\}_{m\in M}$ defined by $E_m = |m\rangle\langle m|$, achieves equality in equation (1).

Conclude that when $\rho$ is invertible,

$$F(\rho, \sigma) = \min_{\{E_m\}_{m\in M}} F(\tilde{\rho}_M, \tilde{\sigma}_M) \tag{4}$$

where the minimum is over POVMs $\{E_m\}_{m\in M}$ and $\tilde{\rho}_M$ and $\tilde{\sigma}_M$ are defined in (2).



4. Given two quantum states $\rho$ and $\sigma$ such that $\rho$ is invertible, use equation (4) to show that

$$D(\rho, \sigma) \geq 1 - F(\rho, \sigma). \tag{5}$$

**Exercise 2** Let $|\psi\rangle_{ABE}$ be a pure state of a tripartite system $ABE$. define the *coherent information* from $A$ to $B$ of $\psi$ to be

$$I_c^{A>B}(\psi) = -S(A|B)_\psi.$$

Here $S(A|B)_\psi$ denotes the conditional entropy of the subsystem $A$ with respect to subsystem $B$, given that the composite system $ABE$ is in the pure state $|\psi\rangle_{ABE}$. Henceforth we shall omit $\psi$.

Prove the following identities:

1. $\frac{1}{2}I(A:B) + \frac{1}{2}I(A:E) = S(A)$
2. $\frac{1}{2}I(A:B) - \frac{1}{2}I(A:E) = I_c^{A>B}$

**Exercise 3** A bipartite quantum state $\rho_{AB}$ is said to be *separable* if it can be written as a convex combination of product states, i.e., if there exists an ensemble $\{p_i, \sigma_A^{(i)} \otimes \tau_B^{(i)}\}$, with $\sigma_A^{(i)} \in \mathcal{B}(\mathcal{H}_A)$ and $\tau_B^{(i)} \in \mathcal{B}(\mathcal{H}_B)$, such that

$$\rho_{AB} = \sum_i p_i \sigma_A^{(i)} \otimes \tau_B^{(i)}.$$

This allows us to extend the definition of entanglement to mixed states: *a mixed state is entangled if it is not separable.*

Show that if $\rho_{AB}$ is separable then $I_c^{A>B} \leq 0$.

What implication does this have on the conditional entropy $S(A|B)$?

**Exercise 4** Use the HSW theorem to find the product state capacity of the depolarizing channel, $\Lambda$, defined by

$$\Lambda(\rho) = p\rho + (1-p)\frac{\mathbf{I}}{2}.$$

**Exercise 5** Alice prepares a photon in one of two polarization states, given by the kets $|a\rangle := |1\rangle$, and $|b\rangle := \sin\theta|0\rangle + \cos\theta|1\rangle$, depending on the outcome of a fair coin toss. If the outcome is heads, she prepares the state $|a\rangle$. Otherwise she prepares the state $|b\rangle$. Evaluate the Holevo $\chi$ quantity for her ensemble of states. (Use the convention that $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ \ 1)^T$). For what value of $\theta$ is the Holevo bound achieved? Explain why.

**Exercise 6** Alice encodes classical information into $n$ photons which she sends to Bob through a quantum channel. What is the maximum number of bits of information that Bob can infer from the output of the channel by doing measurements on it?

*Hint: Use the Holevo bound.*

**Exercise 7 Entropy exchange** The entropy exchange for a state $\rho$ and a quantum channel $\Lambda$ is defined as follows:

$$S(\rho, \Lambda) := S(\rho'_{RQ}),$$

where $\rho'_{RQ} = (\mathrm{id}_R \otimes \Lambda)\psi^\rho_{RQ}$, with $\psi^\rho_{RQ}$ being a purification of $\rho$.

(i) Prove that $S(\rho, \Lambda) = S(\rho'_E$, where $\rho'_E = \mathrm{tr}_{RQ}(\rho'_{RQE})$ with

$$\rho'_{RQE} = \mathrm{tr}_E\left((I_R \otimes U_{QE})(\psi^\rho_{RQ} \otimes |0_E\rangle\langle 0_E|)(I_R \otimes U_{QE})\right),$$

with $U_{QE}$ being the Stinespring dilation of the channel $\Lambda$.

*Thus the entropy exchange can be interpreted as the amount of entropy introduced by $\Lambda$ into an initially pure environment.*

(ii) Prove that the entropy exchange can also be written in the form

$$S(\rho, \Lambda) = S(W) = -\,\mathrm{tr}\, W \log W,$$

where $W$ denotes a matrix with elements $W_{ij} = \mathrm{tr}(A_i \rho A_j)$, where $\{A_i\}$ denote a set of Kraus operators of $\Lambda$.

**Exercise 8 Quantum Fano inequality** Prove the quantum Fano inequality:

$$S(\rho, \Lambda) \leq h(F_e(\rho, \Lambda)) + (1 - F_e(\rho, \Lambda)) \log(d^2 - 1) \tag{6}$$

where



- $\Lambda$ denotes a quantum operation with Kraus representation

$$\Lambda(\rho) = \sum_{i=0}^{d^2} V_i \rho V_i^\dagger,$$

  with $\rho$ being the state of a quantum system $Q$ with Hilbert space of dimension $d$.

- $h(\cdot\,)$ denotes the binary Shannon entropy, i.e., for any $0 < p < 1$,

$$h(p) = -p \log p - (1 - p) \log(1 - p).$$

-

$$S(\rho, \Lambda) = -\,\mathrm{tr}(W \log W),$$

  with $W$ being a matrix with elements $W_{ij} = \mathrm{tr}(V_i \rho V_j)$. $S(\rho, \Lambda)$ is called *entropy exchange*.

-

$$F_e(\rho, \Lambda) := \langle \Psi^\rho_{RQ}|(\mathrm{id} \otimes \Lambda)\Psi^\rho_{RQ}|\Psi^\rho_{RQ}\rangle,$$

  is the entanglement fidelity. Here $|\Psi^\rho_{RQ}\rangle$ is a purification of the state $\rho$, with $R$ denoting the reference system used for the purification.

  *What is the implication of the quantum Fano inequality as regards entanglement?*

**Exercise 9** Continuity of the von Neumann entropy (Fannes' inequality): Suppose $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ are states such that their trace distance $D(\rho, \sigma)$ satisfies the bound

$$2D(\rho, \sigma) = ||\rho - \sigma||_1 \leq 1/e. \tag{7}$$

Then

$$|S(\rho) - S(\sigma)| \leq ||\rho - \sigma||_1 \log d + \eta\left(||\rho - \sigma||_1\right), \tag{8}$$

where $d = \dim\mathcal{H}$, and $\eta(x) := -x \log x$.

Let us prove this theorem in steps:

(i) Let $r_1 \geq r_2 \geq \ldots \geq r_d$ be the eigenvalues of $\rho$ arranged in non-increasing order, and let $s_1 \geq s_2 \geq \ldots \geq s_d$ be the eigenvalues of $\sigma$ arranged in non-increasing order. Then prove that:

$$||\rho - \sigma||_1 \geq \sum_{j=1}^{d} |r_j - s_j| \tag{9}$$

(ii) Check (using elementary calculus) that if $|r-s| \leq 1/2$, then

$$|\eta(r) - \eta(s)| \leq \eta(|r-s|), \tag{10}$$

where $\eta(x) := -x\log x$.

(iii) Use (ii) and the triangle inequality to prove that

$$|S(\rho) - S(\sigma)| \leq \sum_j \eta(|r_j - s_j|), \tag{11}$$

(iv) Let $\varepsilon_j := |r_j - s_j| \ \forall\ j = 1, 2, \ldots, d$, and $\varepsilon := \sum_j \varepsilon_j$. Let $\lambda_j := \varepsilon_j/\varepsilon$ and note that $\{\lambda_j\}$ forms a probability distribution. Use this fact and (iii) to prove that

$$|S(\rho) - S(\sigma)| \leq \varepsilon \log d + \eta(\varepsilon). \tag{12}$$

(v) Note that $\eta(\varepsilon)$ is a monotonically increasing function of $\varepsilon$ for $0 \leq \varepsilon \leq 1/e$. Use this to finally arrive at the statement (8).

**Exercise 10** An interesting class of quantum channels are the *entanglement-breaking (EB) channels*. An EB channel $\Lambda$ is one for which $(I \otimes \Lambda)(\omega)$ is separable, even for entangled $\omega$[1]. The Holevo capacity has been proved to be additive for EB channels.

(i) Prove that any channel of the following form is EB:

$$\Lambda(\rho) = \sum_k \sigma_k \operatorname{tr}(E_k \rho), \tag{13}$$

where $\sigma_k$ are density matrices and $\{E_k\}$ is a POVM. The above form has the following physical interpretation. Alice does a measurement (POVM) on the input state $\rho$ and communicates the outcomes $k$ to Bob via a classical channel; Bob then prepares an agreed upon state $\sigma_k$. Hence, EB channels are also called "measure-and-prepare channels".

(ii) Prove that if the Choi state $(I \otimes \Lambda)|\tilde{\Phi}\rangle\langle\tilde{\Phi}|$ (where $|\tilde{\Phi}\rangle$ denotes the unnormalized maximally entangled state) is separable, then $\Lambda$ has the form (13).

[1]It derives its name from the fact that the channel outputs a separable state whenever half of an entangled state is input to it