QIT Revision Class

University of Cambridge

Part III, Lent 2018

Problem 1.

1. Define the trace distance $D(\rho, \sigma)$ between two states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ and prove that it can be expressed in the form:
$$D(\rho, \sigma) = \frac{1}{2}(\operatorname{tr} Q + \operatorname{tr} R)$$
where Q and R are suitably defined positive semi-definite operators in $\mathcal{B}(\mathcal{H})$.
2. Using the above identity, prove that
$$D(\rho, \sigma) = \max_P \operatorname{tr}(P(\rho - \sigma))$$
where the maximisation is over all projection operators $P \in \mathcal{B}(\mathcal{H})$.
3. Further, prove that
$$D(, \rho, \sigma) = \max_T \operatorname{tr}(T(\rho - \sigma))$$
where the maximisation is over all positive semi-definite operators $T \in \mathcal{B}(\mathcal{H})$ with eigenvalues less than or equal to unity.
4. Let ρ be a quanutm state and Λ be a linear completely positive trace-preserving map. Prove that
$$F_e(\rho, \Lambda) \leq (F(\rho, \Lambda(\rho)))^2$$
where $F_e(\rho, \Lambda)$ denotes the entanglement fidelity, and $F(\rho, \Lambda(\rho))$ denotes the fidelity of the states ρ and $\Lambda(\rho)$.

Problem 2.

1. State the Holevo-Schumacher-Westmoreland theorem.
   (a) Use it to obtain the product-state classical capacity of a qubit depolarizing channel Λ defined as follows:
   $$\Lambda(\rho) = p\rho + \frac{1-p}{3}(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z),$$
   where σ_x, σ_y, and σ_z are the Pauli matrices.
   (b) Consider an ensemble of quantum states $\mathcal{E} = \{p_x, \rho_x\}$ and let $\chi(\mathcal{E})$ denote its Holevo quantity. Let Λ be a quantum channel. Prove that
   $$\chi(\mathcal{E}') \leq \chi(\mathcal{E})$$
   where $\mathcal{E}' = \{p_x, \Lambda(\rho_x)\}$.
2. Consider a memoryless quantum information source characterized by $\{\pi, \mathcal{H}\}$, where $\pi \in \mathcal{D}(\mathcal{H})$. Suppose on n uses, the source emits a signal state $|\Psi_k^{(n)}\rangle \in \mathcal{H}^{\otimes n}$ with probability $p_k^{(n)}$, the index k labelling the different possible signal states. State the Typical Subspace Theorem, and use it prove that for such a source there exists a reliable compression-decompression scheme of rate $R > S(\pi)$ where $S(\pi)$ denotes the von Neumann entropy of the source.

Note: we won't cover part (2) in the revision class, since it's just bookwork.

Problem 3.

1. The Bell states $|\Phi^+_{AB}\rangle$, $|\Phi^-_{AB}\rangle$, $|\Psi^+_{AB}\rangle$, $|\Psi^-_{AB}\rangle$, can be characterized by two classical bits, namely the parity bit and the phase bit. Show that the latter are eigenvalues of two commuting observables.
2. The Bell states form an orthonormal basis of the two-qubit Hilbert space. It is referred to as the Bell basis. Let us denote it by B_1. A sequence of two operations can be used to convert states of the computational basis $B_2 := \{|ij\rangle : i, j \in \{0, 1\}\}$ to the Bell states. State what these operations are. Can they also be used to convert states of B_1 to B_2? Justify your answer.
3. Prove that the Schmidt rank of a pure state *cannot* be increased by local operations and classical communication (LOCC), clearly stating any theorem that you use.
4. It is known that a matrix A is doubly stochastic if and only if $\vec{x} \prec \vec{y}$ for all vectors $\vec{y}$, where $x = A\vec{y}$.
   Let $\rho \in \mathcal{D}(\mathcal{H})$ be a state, where $\dim \mathcal{H} = d$, and let $\Lambda : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ be a *unital* channel. Let $\vec{r} = (r_1, r_2, \dots, r_d)$ and $\vec{s} = (s_1, s_2, \dots, s_d)$ respectively denote the vectors of eigenvalues of ρ and $\sigma = \Lambda(\rho)$, arranged in non-increasing order. Using the above result, prove that $\vec{r} \prec \vec{s}$.

Problem 4.

1. Prove that any completely positive trace-preserving map Φ, acting on states ρ in a Hilbert space $\mathcal{H}_A$ can be written in the Kraus form:
   $$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger,$$
   where A_k are linear operators satisfying $\sum_k A_k^\dagger A_k = I$ and I is the identity operator.

   Hint: consider a maximally entangled state.
2. Using a maximally entangled state and the properties of the swap operator, prove that the transposition operator T is positive but not completely positive.

Problem 5.

Consider the decay of a two-level atom from its excited state to its ground state. Let the probability of this decay be p. The spontaneous emission of a photon accompanies this decay.

1. Name the quantum channel that can be used to model this process and write its Kraus operators. What process does each of these Kraus operators correspond to? Give reasons for your answer.
2. What is a unital channel? Is the channel in (1) unital?
3. Suppose the atom is originally in a state $\rho := \sum_{\alpha,\beta=0}^{1} \rho_{\alpha\beta} |\alpha\rangle\langle\beta|$, where $|\alpha\rangle, |\beta\rangle$, $\alpha, \beta \in \{0, 1\}$, denote orthonormal basis states of the Hilbert space of the atom. By considering the action of a unitary operator U on the atom and its environment, deduce how the state of the atom changes under the action of one use of the channel in (1).
4. Let AB denote a bipartite system which is in state ρ_{AB}. Show that the mutual information of the system cannot increase under the action of a completely positive trace-preserving map on the subsystem B alone.