

Part III, Lent 2019  
Quantum Information Theory

**Example Sheet 2**

**Exercise 1**

1. Prove that if  $A$  is positive semi-definite, then  $A$  is Hermitian.
2. The action of a flip operator  $\mathbb{F} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , where  $\mathcal{H}_A, \mathcal{H}_B \simeq \mathbf{C}^d$ , is defined through the following equation:

$$\mathbb{F}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle,$$

where  $\{|i\rangle\}_{i=1}^d$  denotes an orthonormal basis of  $\mathbf{C}^d$ . Write an explicit expression for  $\mathbb{F}$  in terms of the basis vectors  $|i\rangle$ . What are the eigenvalues of  $\mathbb{F}$  and what are their multiplicities? Express  $\mathbb{F}$  in terms of the MES

$$|\Omega\rangle := \frac{1}{\sqrt{d}}|i\rangle \otimes |i\rangle.$$

*Hint: Think of what mathematical operation you would use to relate one to the other.*

Use the relation that you obtain to rewrite equations (1.4) and (1.5) of Lemma 1 of Notes 7 (on the course webpage) using the flip operator instead of the MES.

**Solution**

1. If  $A \geq 0$ , then  $\langle v, Av \rangle \geq 0$  for all  $|v\rangle \in \mathcal{H}$ . In particular,  $\langle v|Av\rangle \in \mathbb{R}$ . Then
 
$$\langle v|Av\rangle = \overline{\langle v|Av\rangle} = \langle Av|v\rangle = \langle v|A^*v\rangle.$$

That is,

$$\langle v|(A - A^*)v\rangle = 0, \quad \forall |v\rangle \in \mathcal{H}. \quad (1)$$

Next, we can see that the operator  $B := A - A^*$  is *normal*, i.e. it commutes with its adjoint. Since  $B^* = A^* - A = -B$ , we have

$$BB^* = B(-B) = -B^2$$

and

$$B^*B = (-B)B = -B^2.$$

Thus,  $[B, B^*] = 0$  and thus  $B$  is normal. Therefore,  $B$  is diagonal in some basis  $|i\rangle$ . Since  $\langle i|B|i\rangle$  is the  $i$ th entry on the diagonal when  $B$  is written in the basis  $|i\rangle$ , by letting  $|v\rangle$  range over  $|i\rangle$  in (1), we see that all of the diagonal entries of  $B$  are zero. Since all the other entries of  $B$  are zero in this basis too,  $B$  is thus the zero matrix. Therefore,  $A = A^*$ .

2. Since  $\mathbb{F}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle$ , we see that we can write

$$\mathbb{F} = \sum_{i,j} (|j\rangle \otimes |i\rangle)(\langle i| \otimes \langle j|) = \sum_{i,j} |ji\rangle \langle ij|. \quad (2)$$

Indeed, this must be a matrix representation of  $\mathbb{F}$  as it agrees with  $\mathbb{F}$  on the basis  $\{|i\rangle \otimes |j\rangle\}$ . We can see that  $\mathbb{F}$  is self-adjoint:

$$\langle ij, \mathbb{F}ij\rangle = \langle ij, ji\rangle = \delta_{ij} = \langle ji, ij\rangle = \langle \mathbb{F}ij, ij\rangle = \langle ij, \mathbb{F}^*ij\rangle. \quad (3)$$

That is,  $\langle ij, (\mathbb{F} - \mathbb{F}^*)ij\rangle = 0$  for all  $i, j$ . Since  $\{|i\rangle \otimes |j\rangle\}_{i,j}$  is a basis of  $\mathbf{C}^d \otimes \mathbf{C}^d$ , we have  $\langle v, (\mathbb{F} - \mathbb{F}^*)v\rangle = 0$  for every  $|v\rangle \in \mathbf{C}^d \otimes \mathbf{C}^d$ . Just as before, we find we must have  $\mathbb{F} = \mathbb{F}^*$ .

We can see that  $\mathbb{F}^2 = \mathbb{1}$  since

$$\mathbb{F}^2|i\rangle \otimes |j\rangle = \mathbb{F}|j\rangle \otimes |i\rangle = |i\rangle \otimes |j\rangle. \quad (4)$$

Since the eigenvalues of  $\mathbb{F}^2$  are the squares of the eigenvalues of  $\mathbb{F}$ , we must have that every eigenvalue of  $\mathbb{F}$  is  $\pm 1$ . Moreover, they must not all be 1 since  $\mathbb{F} \neq \mathbb{1}$ .

The eigenspace associated to  $+1$  is the so-called ‘‘symmetric subspace’’: the subspace of vectors of  $\mathbf{C}^d \otimes \mathbf{C}^d$  which is invariant under swapping, while the eigenspace associated to  $-1$  is the ‘‘anti-symmetric subspace’’.

Define

$$P_{\text{sym}} = \frac{1}{2}(\mathbb{1} + F), \quad P_{\text{asym}} = \frac{1}{2}(\mathbb{1} - F). \quad (5)$$

$$P_{\text{sym}}^2 = \frac{1}{4}(\mathbb{1} + 2\mathbb{F} + \mathbb{F}^2) = \frac{1}{4}(2\mathbb{1} + 2\mathbb{F}) = P_{\text{sym}}. \quad (6)$$

Since  $\mathbb{1}$  and  $\mathbb{F}$  are self-adjoint, so is  $P_{\text{sym}}$ . Thus,  $P_{\text{sym}}$  is an orthogonal projection, and thus positive semi-definite. Similarly, we see that

$$P_{\text{asym}}^2 = \frac{1}{4}(\mathbb{1}^2 + F^2 - 2F) = \frac{1}{2}(\mathbb{1} - F) = P_{\text{asym}} \quad (7)$$

and hence  $P_{\text{asym}}$  is an orthogonal projection. Moreover,

$$\mathbb{F} = P_{\text{sym}} - P_{\text{asym}}. \quad (8)$$

We have written  $\mathbb{F}$  as a sum of its eigenvalues times orthogonal projections, and thus have found the spectral decomposition of  $\mathbb{F}$ . The multiplicity of  $+1$  is thus the dimension of the subspace associated to  $P_{\text{sym}}$ , which is

$$\dim \text{span}\{|v\rangle \in \mathbf{C}^d \otimes \mathbf{C}^d : P_{\text{sym}}|v\rangle = |v\rangle\} = \text{rank } P_{\text{sym}} = \text{tr} P_{\text{sym}} \quad (9)$$

using that  $P_{\text{sym}}$  is a orthogonal projection, so its rank is its trace. The trace we may easily compute:

$$\begin{aligned} \text{tr} P_{\text{sym}} &= \frac{d^2}{2} + \frac{1}{2} \text{tr}[\mathbb{F}] \\ &= \frac{d^2}{2} + \frac{1}{2} \sum_{ij} \langle ij | \mathbb{F} | ij \rangle \\ &= \frac{d^2}{2} + \frac{1}{2} \sum_{ij} \langle ij | j \bar{i} \rangle \\ &= \frac{d^2}{2} + \frac{1}{2} \sum_{ij} \delta_{ij} \\ &= \frac{d^2}{2} + \frac{1}{2} d = \frac{d^2 + 1}{2}. \end{aligned}$$

The multiplicity of  $-1$  is therefore  $d^2 - \frac{d^2+1}{2} = \frac{d^2-1}{2}$ , since the total number of eigenvalues is  $d^2$  (since  $\mathbb{F}$  is an operator on a  $d^2$ -dimensional vector space,  $\mathbf{C}^d \otimes \mathbf{C}^d$ ).

To relate  $\mathbb{F}$  to  $|\Omega\rangle$  we note that

$$|\Omega\rangle := |\Omega\rangle \langle \Omega| = \frac{1}{d} \sum_{ij} |i\rangle \langle j| \otimes |i\rangle \langle j|. \quad (10)$$

Thus, if we take the transpose over the second system in the basis  $|i\rangle$  as the operator  $T$ , we have

$$(\text{id} \otimes T) \Omega = \frac{1}{d} \sum_{ij} |i\rangle \langle j| \otimes |j\rangle \langle i| = \frac{1}{d} \mathbb{F}. \quad (11)$$

Thus,

$$\mathbb{F} = d(\text{id} \otimes T) \Omega. \quad (12)$$

Lastly, we wish to rewrite

$$\langle \Omega | A \otimes B | \Omega \rangle = \frac{1}{d} \text{tr}[A^T B] \quad (1.4)$$

and

$$A \otimes I | \Omega \rangle = (I \otimes A^T) | \Omega \rangle \quad (1.5)$$

using  $\mathbb{F}$  instead of  $|\Omega\rangle$ .

First, we note

$$\langle \Omega | A \otimes B | \Omega \rangle = \text{tr}[(A \otimes B) |\Omega\rangle \langle \Omega|] = \text{tr}[(A \otimes B) \Omega]. \quad (13)$$

Then, since  $\Omega = \frac{1}{d}(\text{id} \otimes T) \mathbb{F}$  (since  $T^2 = \text{id}$ ), we have

$$\langle \Omega | A \otimes B | \Omega \rangle = \frac{1}{d} \text{tr}[(A \otimes B)(\text{id} \otimes T) \mathbb{F}]. \quad (14)$$

Therefore, (1.4) becomes

$$\frac{1}{d} \text{tr}[(\text{id} \otimes T)(\mathbb{F})(A \otimes B)] = \frac{1}{d} \text{tr}[A^T B]. \quad (15)$$

Note  $\text{id} \otimes T$  is a superoperator acting on  $\mathbb{F}$ , the result of which is then multiplied by  $(A \otimes B)$  in the trace.

For (1.5), we may consider the action of  $A \otimes I$  on  $\Omega$ . We have

$$(A \otimes I)\Omega = (A \otimes I)|\Omega\rangle\langle\Omega| = (I \otimes A^T)|\Omega\rangle\langle\Omega| = (I \otimes A^T)\Omega. \quad (16)$$

Therefore, using  $\Omega = \frac{1}{\sqrt{d}}(\text{id} \otimes T)\mathbb{F}$ , we have

$$(A \otimes I)(\text{id} \otimes T)\mathbb{F} = (I \otimes A^T)(\text{id} \otimes T)\mathbb{F}. \quad (17)$$

**Exercise 2** Show that any density operator of a qubit can be written as

$$\varrho = \frac{1}{2}(\mathbb{1} + \vec{s} \cdot \vec{\sigma}) = \frac{1}{2} \sum_{k=0}^3 s_k \sigma_k, \quad (18)$$

where  $\vec{s} = (s_1, s_2, s_3)$  is a real three-dimensional vector such that  $\|\vec{s}\| \leq 1$ . In the above  $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ , with  $\sigma_1, \sigma_2, \sigma_3$  being the three Pauli matrices  $\sigma_x, \sigma_y, \sigma_z$ . Moreover,  $\sigma_0$  denotes the  $2 \times 2$  identity matrix.

The vector  $\vec{s}$  is usually referred to as the Bloch vector. The set of all Bloch vectors defines a sphere which is called the *Bloch sphere*. It provides a useful geometrical representation of the states of a qubit.

Show that  $s_k = \text{tr} \sigma_k \varrho$ . Show that  $\|\vec{s}\| = 1$  for a pure state  $\rho$ . What is the Bloch representation of the completely mixed state  $\rho = \mathbf{I}/2$ ? What do the North Pole and the South Pole of the Bloch sphere correspond to?

### Solution

Any qubit density matrix can be written as

$$\rho = \begin{pmatrix} a & b - ic \\ b + ic & 1 - a \end{pmatrix} \quad (19)$$

where  $a, b, c$  are real numbers. This is simply because  $\rho$  is trace 1 and self-adjoint. For any vector  $\vec{s} = (s_x, s_y, s_z)$ , we can compute

$$\vec{s} \cdot \vec{\sigma} = \begin{pmatrix} s_z & s_x - is_y \\ s_x + is_y & -s_z \end{pmatrix} \quad (20)$$

Therefore,

$$\frac{I}{2} + \frac{1}{2}\vec{s} \cdot \vec{\sigma} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}s_z & \frac{1}{2}s_x - \frac{1}{2}is_y \\ \frac{1}{2}s_x + \frac{1}{2}is_y & \frac{1}{2} - \frac{1}{2}s_z \end{pmatrix} \quad (21)$$

Now, simply choose  $s_z = 2a - 1$ ,  $s_x = 2b$  and  $s_y = 2c$ . By checking each matrix entry, we find that  $\rho = \frac{I}{2} + \frac{1}{2}\vec{s} \cdot \vec{\sigma}$ . To check  $\|\vec{s}\| \leq 1$ , we need to use more properties of  $\rho$ . First,

$$\begin{aligned} \|\vec{s}\|^2 &= s_x^2 + s_y^2 + s_z^2 = (2a - 1)^2 + 4b^2 + 4c^2 \\ &= 4a^2 + 1 - 4a + 4b^2 + 4c^2 \\ &= 1 - 4(a - r) \end{aligned}$$

for  $r := a^2 + b^2 + c^2$ . Therefore,  $\|\vec{s}\| = \sqrt{1 - 4(a - r)}$ .

Since  $\rho$  is a density matrix, its eigenvalues are non-negative. This implies a constraint on  $a, b, c$ . We solve the characteristic equation by setting the determinant of  $\rho - \lambda I$  to zero:

$$\begin{aligned} 0 &= \begin{vmatrix} a - \lambda & b - ic \\ b + ic & 1 - a - \lambda \end{vmatrix} = (a - \lambda)(1 - a - \lambda) - (b^2 + c^2) \\ &\quad \lambda^2 - \lambda + a - (a^2 + b^2 + c^2) = 0. \end{aligned}$$

Then

$$\lambda = \frac{1 \pm \sqrt{1 - 4(a - r)}}{2} \geq 0. \quad (22)$$

That is,

$$1 - \sqrt{1 - 4(a - r)} \geq 0 \iff \sqrt{1 - 4(a - r)} \leq 1. \quad (23)$$

That is,  $\|\vec{s}\| \leq 1$ . Moreover,  $\|\vec{s}\| = 1$  if and only if one of the eigenvalues is zero, by (22). Thus,  $\|\vec{s}\| = 1$  if and only if  $\rho$  is pure. By the same equation, we can see if the two eigenvalues are  $\frac{1}{2}$ , we must have  $\|\vec{s}\| = \sqrt{1 - 4(a - r)} = 0$ . That is, the Bloch vector for the completely mixed state is  $\vec{s} = \vec{0}$ .

The north pole is when  $\vec{s} = (0, 0, 1)$ . In that case, since  $s_z = 2a - 1$ ,  $s_x = 2b$  and  $s_y = 2c$ , we have  $a = 1$ ,  $b = c = 0$ . That is,  $\rho = |0\rangle\langle 0|$ . For the south pole,  $\vec{s} = (0, 0, -1)$ . Then  $a = b = c = 0$ , so  $\rho = |1\rangle\langle 1|$ .

Next,

$$\text{tr}[\sigma_k \rho] = \frac{1}{2} \text{tr}[\sigma_k] + \frac{1}{2} \sum_{j \neq k} s_j \text{tr}[\sigma_k \sigma_j] + \frac{1}{2} s_k \text{tr}[\sigma_k^2]. \quad (24)$$

Note each Pauli matrix is traceless (besides the identity), so the first term is zero. For the second term, we can use the relation

$$\sigma_a \sigma_b = \delta_{ab} I + i \sum_c \varepsilon_{abc} \sigma_c \quad (25)$$

where  $\varepsilon_{abc}$  is either  $+1$ ,  $-1$ , or zero, depending on  $a, b, c$  (this is the so-called Levi-Civita symbol). Taking the trace,  $\text{tr}[\sigma_a \sigma_b] = \delta_{ab} \text{tr}[I] = 2\delta_{ab}$ , since  $\sigma_c$  is traceless. Finally, for any Pauli matrix,  $\sigma_k^2 = I$ . So (24) becomes

$$\text{tr}[\sigma_k \rho] = \frac{1}{2} \cdot 0 + \frac{1}{2} \sum_{j \neq k} s_j \cdot 0 + \frac{1}{2} s_k (2) = s_k. \quad (26)$$

**Exercise 3** Given the density matrix

$$\rho = \begin{pmatrix} \frac{3}{5} & \frac{1}{4} - i\frac{1}{6} \\ \frac{1}{4} + i\frac{1}{6} & \frac{2}{5} \end{pmatrix}$$

what is the Bloch vector  $\vec{s} = (s_x, s_y, s_z)$  for  $\rho$ ? Is it a pure state or a mixed state? What is the probability that a measurement of the spin of the qubit along the  $Z$ -axis will yield a value  $+1$ ?

**Solution**

We can write

$$\rho = \frac{1}{2}I + \frac{1}{2} \begin{pmatrix} \frac{1}{5} & \frac{1}{2} - \frac{i}{3} \\ \frac{1}{2} + \frac{i}{3} & -\frac{1}{5} \end{pmatrix} \quad (27)$$

Note

$$\vec{s} \cdot \vec{\sigma} = \begin{pmatrix} s_z & s_x - is_y \\ s_x + is_y & -s_z \end{pmatrix} \quad (28)$$

Therefore, we can identify  $s_x = \frac{1}{2}$ ,  $s_y = \frac{1}{3}$ , and  $s_z = \frac{1}{5}$ . Since

$$\|\vec{s}\| = \sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{25}} = \frac{19}{30} < 1, \text{ the state } \rho \text{ is mixed.}$$

Since

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (29)$$

the projector onto its  $+1$  eigenspace is just  $|0\rangle\langle 0|$ . Therefore, the probability of getting outcome  $+1$  when measuring  $\rho$  by the  $\sigma_z$  observable is given by

$$\text{tr}[\rho \sigma_z] = |0\rangle\langle 0| \rho |0\rangle\langle 0| = \frac{1}{2} + \frac{1}{2} \frac{1}{5} = \frac{6}{10} = \frac{3}{5}. \quad (30)$$

**Exercise 4** Show that the set of density operators acting on a Hilbert space  $\mathcal{H}$ , where  $\dim \mathcal{H} = d$  is a *convex subset* of the real vector space of  $d \times d$  Hermitian matrices. Show that pure states are extremal points of this set.

**Solution**

Let  $\rho, \sigma$  be density operators. To show the set of density operators is convex, we only need to show that  $t\rho + (1-t)\sigma$  is a density operator for every  $t \in (0, 1)$ . Note  $t\rho + (1-t)\sigma$  is called a *convex combination* of  $\rho$  and  $\sigma$ , if  $t \in (0, 1)$ . We have

$$\text{tr}[t\rho + (1-t)\sigma] = t\text{tr}[\rho] + (1-t)\text{tr}[\sigma] = t + 1 - t = 1. \quad (31)$$

Additionally, for every vector  $|v\rangle \in \mathcal{H}$ ,

$$\langle v, [t\rho + (1-t)\sigma]v \rangle = t \langle v, \rho v \rangle + (1-t) \langle v, \sigma v \rangle. \quad (32)$$

Since  $\rho$  and  $\sigma$  are positive semidefinite,  $\langle v, \rho v \rangle \geq 0$  and  $\langle v, \sigma v \rangle \geq 0$ . As the sum of two positive numbers, we then have  $\langle v, [t\rho + (1-t)\sigma]v \rangle \geq 0$ . Since  $|v\rangle \in \mathcal{H}$  is arbitrary, we therefore have that  $t\rho + (1-t)\sigma$  is positive semi-definite. Thus,  $t\rho + (1-t)\sigma$  is a density matrix, proving that the set of density matrices is convex.

A point  $x$  in a convex set  $\mathcal{C}$  is called extremal if  $\lambda y + (1-\lambda)z = x$  for  $y, z \in \mathcal{C}$  and  $\lambda \in (0, 1)$  implies that  $x = y = z$ . Consider therefore  $|\psi\rangle = \lambda\rho_1 + (1-\lambda)\rho_2$  where  $|\psi\rangle \in \mathcal{H}$  is a pure state and  $\lambda \in (0, 1)$ . Let  $|\varphi\rangle \in \text{span}\{|\psi\rangle\}^\perp$  be any vector from the orthogonal complement to the subspace spanned by  $|\psi\rangle$ . Then we have

$$0 = \langle \varphi | \psi \rangle \langle \varphi | = \lambda \langle \varphi | \rho_1 | \varphi \rangle + (1-\lambda) \langle \varphi | \rho_2 | \varphi \rangle.$$

Since  $\lambda > 0$  and  $1-\lambda > 0$ , this implies  $\langle \varphi | \rho_1 | \varphi \rangle = \langle \varphi | \rho_2 | \varphi \rangle = 0$  for all  $|\varphi\rangle \in \text{span}\{|\psi\rangle\}^\perp$ . Hence,  $\rho_1 = \rho_2 = \psi$ . This shows that every pure state is extremal.

On the other hand, assume  $\rho$  is extremal and not pure; we will derive a contradiction. We may write  $\rho = \sum_{i=0}^{d-1} \lambda_i |\psi_i\rangle\langle \psi_i|$  the spectral decomposition of  $\rho$ ; since  $\rho$  is not pure, there are at least two non-zero  $\lambda_i$ , so assume  $\lambda_0$  and  $\lambda_1$  are non-zero. Then

$$\begin{aligned} \rho &= \lambda_0 |\psi_0\rangle\langle \psi_0| + \sum_{i=1}^{d-1} \lambda_i |\psi_i\rangle\langle \psi_i| \\ &= \lambda_0 |\psi_0\rangle\langle \psi_0| + (1-\lambda_0) \sum_{i=1}^{d-1} \frac{\lambda_i}{1-\lambda_0} |\psi_i\rangle\langle \psi_i|. \end{aligned}$$

Since  $\sum_{i=1}^{d-1} \frac{\lambda_i}{1-\lambda_0} = \frac{1-\lambda_0}{1-\lambda_0} = 1$ , taking  $\rho_1 = |\psi_0\rangle\langle\psi_0|$  and  $\rho_2 = \sum_{i=1}^{d-1} \frac{\lambda_i}{1-\lambda_0} |\psi_i\rangle\langle\psi_i|$ , we have written  $\rho = \lambda_0\rho_1 + (1-\lambda_0)\rho_2$  for some  $\lambda_0 \in (0, 1)$  for density matrices  $\rho_1$  and  $\rho_2$ . Since  $\rho$  was assumed to be extremal, we must have  $\rho = \rho_1 = \rho_2$ . Since  $\rho_1$  is a pure state, we must have  $\rho$  pure too.

**Exercise 5** Find the Schmidt ranks for each of the following states

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |44\rangle) & |\phi_2\rangle &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\ |\phi_3\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) & |\phi_4\rangle &= \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle) \end{aligned}$$

**Solution**

$|\phi_1\rangle$  is already in Schmidt form,

$$|\phi_1\rangle = \sum_{i=1}^4 \frac{1}{4} |ii\rangle. \quad (33)$$

For the state  $|\phi_2\rangle$ , we see

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \quad (34)$$

is a separable pure state, and therefore has Schmidt rank 1.

The state  $|\phi_3\rangle$  is a maximally entangled state, as can be verified by hand, and thus has Schmidt rank 2. A shorter proof proceeds as follows. We may write

$$|\phi_3\rangle = \frac{1}{2} |0\rangle (|0\rangle + |1\rangle) + \frac{1}{2} |1\rangle (|0\rangle - |1\rangle). \quad (35)$$

Often we use the notation

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \quad (36)$$

Note  $\{|+\rangle, |-\rangle\}$  is a basis of  $\mathbf{C}^2$ . Then (35) shows

$$|\phi_3\rangle = \frac{1}{\sqrt{2}} |0\rangle |+\rangle + \frac{1}{\sqrt{2}} |1\rangle |-\rangle \quad (37)$$

which is a Schmidt decomposition of  $|\phi_3\rangle$ , which is therefore of Schmidt rank 2.

The fourth state is  $|\phi_4\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle)$ . Then

$$\begin{aligned} |\phi_4\rangle\langle\phi_4| &= \frac{1}{3}[|00\rangle\langle 00| + |00\rangle\langle 01| + |00\rangle\langle 11| \\ &\quad |01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 11| \\ &\quad |11\rangle\langle 00| + |11\rangle\langle 01| + |11\rangle\langle 11|]. \end{aligned}$$

Then

$$\begin{aligned} \text{tr}_B |\phi_4\rangle\langle\phi_4| &= \frac{1}{3}[|0\rangle\langle 0| \\ &\quad |0\rangle\langle 0| + |0\rangle\langle 1| \\ &\quad + |1\rangle\langle 0| + |1\rangle\langle 1|] \\ &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

This has eigenvalues  $\frac{3 \pm \sqrt{5}}{2}$ . Since  $\text{tr}_B |\phi_4\rangle\langle\phi_4|$  has exactly two non-zero eigenvalues, the Schmidt rank of  $|\phi_4\rangle$  is two.

**Exercise 6** Let  $|\Psi_{AR}\rangle$  and  $|\Phi_{AR}\rangle$  be two purifications of a state  $\rho$  of a system  $A$  to a composite system  $AR$ . Prove that there exists a unitary transformation  $U_R$  acting on system  $R$  alone such that

$$|\Phi_{AR}\rangle = (\mathbf{I}_A \otimes U_R)|\Psi_{AR}\rangle.$$

**Solution**

We have

$$\begin{aligned} |\Phi_{AR}\rangle &= (R \otimes I)|\Omega\rangle = (I \otimes R^T)|\Omega\rangle \\ |\Psi_{AR}\rangle &= (R' \otimes I)|\Omega\rangle = (I \otimes R'^T)|\Omega\rangle \end{aligned}$$

for some  $R, R'$ . Thus,

$$\rho = \text{tr}_R \Phi_{AR} = R \frac{I}{d} R^\dagger = \text{tr}_R \Psi_{AR} = R' \frac{I}{d} R'^\dagger$$

and hence  $RR^\dagger = R'R'^\dagger$ . By the Polar decomposition,

$$R = \sqrt{RR^\dagger}U$$

for some unitary  $U$ , and likewise

$$R' = \sqrt{R'R'^\dagger}U'$$

so

$$R'U'^\dagger = \sqrt{R'R'^\dagger} = \sqrt{RR^\dagger} = RU^\dagger$$

and hence

$$R' = RU^\dagger U'.$$

Thus,  $R'^T = (U^\dagger U')^T R^T$ . Since  $V = (U^\dagger U')^T$  is a unitary, we have

$$|\Psi_{AR}\rangle = (I \otimes R'^T) |\Omega\rangle = (I \otimes V R^T) |\Omega\rangle = (I \otimes V) |\Phi_{AR}\rangle$$

as desired.

### Exercise 7 (Surjectivity of $\Lambda \mapsto J$ )

The Choi-Jamolkowski matrix  $J \in \mathcal{B}(\mathbf{C}^{d'} \otimes \mathbf{C}^d)$  and a linear map  $\Lambda : \mathcal{M}_d \rightarrow \mathcal{M}_{d'}$  are related by

$$\begin{aligned} J &= (\Lambda \otimes \text{id}_d)(|\Omega\rangle\langle\Omega|) \\ \text{tr}[A\Lambda(B)] &= d \text{tr}[J(A \otimes B^T)] \end{aligned}$$

for any  $A \in \mathcal{M}_{d'}$  and  $B \in \mathcal{M}_d$ , where  $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle |i\rangle$ . Prove that the map  $\Lambda \mapsto J$  is surjective.

*Hint: Decompose  $J$  into a linear combination of rank-1 operators  $|\psi_i\rangle\langle\psi_j|$  and use  $|\psi_i\rangle = (R_i \otimes I) |\Omega\rangle$ .*

### Solution

Let  $J$  be any matrix  $J \in \mathcal{B}(\mathbf{C}^{d'} \otimes \mathbf{C}^d)$ . We need to prove there is a linear map  $\Lambda$  so that  $J = J(\Lambda)$ .

We may write  $J = \sum_{i,j} \alpha_{ij} |\psi_i\rangle\langle\psi_j|$ . Since  $|\psi_i\rangle = (R_i \otimes I) |\Omega\rangle$  for some  $R_i$ , we have

$$J = \sum_{i,j} \alpha_{ij} (R_i \otimes I) |\Omega\rangle\langle\Omega| (R_j^\dagger \otimes I).$$

Then  $\Lambda(A) = \sum_{i,j} \alpha_{ij} R_i A R_j^\dagger$  is a linear map  $\Lambda : \mathcal{M}_d \rightarrow \mathcal{M}_{d'}$ . Moreover, we immediately have  $J(\Lambda) = J$  as desired.

### Exercise 8 (Unitary freedom in the Kraus decomposition)

Suppose  $\{A_i\}_{i=1}^n$  and  $\{B_i\}_{i=1}^n$  are Kraus operators giving rise to quantum operations  $\Lambda$  and  $\Lambda'$ , respectively. Prove that  $\Lambda = \Lambda'$  if and only if there exist complex numbers  $u_{ij}$  such that  $A_i = \sum_{j=1}^n u_{ij} B_j$ , and  $u_{ij}$  are the elements of an  $n \times n$  unitary matrix.

### Solution

Assume  $A_i = \sum_{j=1}^n u_{ij} B_j = \sum_{j=1}^n \langle i|Uj\rangle B_j$ . Then for any operator  $X \in \mathbf{B}(\mathcal{H})$ ,

$$\begin{aligned} \Lambda(X) &= \sum_i A_i X A_i^* \\ &= \sum_{i,j,k=1}^n \langle i|Uj\rangle \overline{\langle i|Uk\rangle} B_j X B_k^* \\ &= \sum_{i,j,k=1}^n \langle i|Uj\rangle \langle Uk,i\rangle B_j X B_k^* \\ &= \sum_{i,j,k=1}^n \langle i|Uj\rangle \langle k,U^*i\rangle B_j X B_k^* \\ &= \sum_{i,j,k=1}^n \langle k,U^*i\rangle \langle i|Uj\rangle B_j X B_k^* \\ &= \sum_{j,k=1}^n \langle k,U^*Uj\rangle B_j X B_k^* \\ &= \sum_{j,k=1}^n \langle k,j\rangle B_j X B_k^* \\ &= \sum_{j=1}^n B_j X B_j^* \\ &= \Lambda'(X). \end{aligned}$$

On the other hand, if  $\Lambda = \Lambda'$ , then

$$(\text{id} \otimes \Lambda)\Omega = (\text{id} \otimes \Lambda')\Omega \tag{38}$$

for  $\Omega = |\Omega\rangle\langle\Omega|$ , the maximally entangled state,  $|\Omega\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |ii\rangle$ . That is,

$$(\text{id} \otimes \Lambda)\Omega = \sum_{i=1}^n (\mathbb{1} \otimes A_i) |\Omega\rangle\langle\Omega| (\mathbb{1} \otimes A_i^*).$$

Setting  $|\psi_i\rangle = (\mathbb{1} \otimes A_i) |\Omega\rangle$ , we see

$$(\text{id} \otimes \Lambda)\Omega = \sum_i |\psi_i\rangle\langle\psi_i|. \quad (39)$$

Likewise,

$$(\text{id} \otimes \Lambda')\Omega = \sum_{i=1}^n (\mathbb{1} \otimes B_i) |\Omega\rangle\langle\Omega| (\mathbb{1} \otimes B_i^*)$$

so setting  $|\tilde{\psi}_i\rangle = (\mathbb{1} \otimes B_i) |\Omega\rangle$ , we have

$$(\text{id} \otimes \Lambda')\Omega = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|. \quad (40)$$

Therefore,  $\Lambda = \Lambda'$  implies

$$\rho := \frac{1}{n} \sum_i |\psi_i\rangle\langle\psi_i| = \frac{1}{n} \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|. \quad (41)$$

That is, we have two decompositions of a state  $\rho$  into pure states. We can purify  $\rho$ :

$$\rho = \text{tr}_C |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle = \sum_j |\psi_j\rangle \otimes |j\rangle_C. \quad (42)$$

We can construct another purification from the other decomposition, namely

$$\rho = \text{tr}_C |\Phi\rangle\langle\Phi|, \quad |\Phi\rangle = \sum_j |\tilde{\psi}_j\rangle \otimes |j\rangle_C. \quad (43)$$

Therefore, by the previous question, for some unitary  $U_C$ , we have

$$|\Psi\rangle = \mathbb{1} \otimes U_C |\Phi\rangle. \quad (44)$$

That is,

$$\sum_j |\psi_j\rangle \otimes |j\rangle_C = \sum_j |\tilde{\psi}_j\rangle \otimes U_C |j\rangle_C. \quad (45)$$

Since these two states are equal, the results of acting on them by the linear operator  $\text{id} \otimes \langle k|$  must be equal, for any vector  $|k\rangle$ . Note we can see  $\text{id} \otimes \langle k|$  as shorthand for the linear map

$$|\chi\rangle \otimes |\chi'\rangle \mapsto \langle k|\chi'\rangle |\chi\rangle \quad (46)$$

defined for any vectors  $|\chi\rangle, |\chi'\rangle$  which we define on all of  $\mathbf{C}^d \otimes \mathbf{C}^d$  by extending by linearity.

Hence,

$$\sum_j |\psi_j\rangle \langle k|j\rangle_C = \sum_j |\tilde{\psi}_j\rangle \langle k|U_C |j\rangle_C. \quad (47)$$

Since  $\langle k|j\rangle_C = \delta_{kj}$ ,

$$|\psi_k\rangle = \sum_j \langle k|U_C |j\rangle |\tilde{\psi}_j\rangle. \quad (48)$$

Writing  $u_{kj} = \langle k|U_C |j\rangle$ , we have

$$|\psi_k\rangle = \sum_j u_{kj} |\tilde{\psi}_j\rangle. \quad (49)$$

Substituting our definitions of  $|\psi_j\rangle$  and  $|\tilde{\psi}_j\rangle$ , we have

$$(\mathbb{1} \otimes A_k) |\Omega\rangle = \sum_j u_{kj} (\mathbb{1} \otimes B_j) |\Omega\rangle. \quad (50)$$

That is,

$$\frac{1}{\sqrt{n}} \sum_\ell (|\ell\rangle \otimes A_k |\ell\rangle) = \frac{1}{\sqrt{n}} \sum_\ell \sum_j u_{kj} (|\ell\rangle \otimes B_j |\ell\rangle). \quad (51)$$

Now, acting by  $\langle \ell| \otimes \mathbb{1}$ , we have

$$\frac{1}{\sqrt{n}} A_k |\ell\rangle = \frac{1}{\sqrt{n}} \sum_j u_{kj} B_j |\ell\rangle. \quad (52)$$

That is, for any basis vector  $|\ell\rangle$ ,

$$A_k |\ell\rangle = \sum_j u_{kj} B_j |\ell\rangle. \quad (53)$$

Thus,

$$A_k = \sum_j u_{kj} B_j. \quad (54)$$

This concludes the proof.

**Exercise 9 (Number of Kraus operators)**

1. Consider a quantum operation  $\Lambda : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$ . What is the maximal rank of its Choi state  $J(\Lambda)$ ?
2. Prove that any quantum operation  $\Lambda$  on the state  $\rho$  of a system with a  $d$ -dimensional Hilbert space, there exists a Kraus decomposition with at most  $d^2$  elements, i.e.,

$$\Lambda(\rho) = \sum_{i=1}^n A_i \rho A_i^\dagger,$$

where  $n \leq d^2$ .

**Solution**

1. Let  $\mathcal{H}_A \cong \mathbf{C}^{d_A}$ . Then for  $|\Omega\rangle = \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |ii\rangle \in \mathcal{H}_A \otimes \mathbf{C}^{d_A}$ , the Choi state of  $\Lambda$  is

$$(\Lambda \otimes \text{id}_{d_A})(\Omega) = \frac{1}{d_A} \sum_{ij} \Lambda(|i\rangle\langle j|) \otimes |i\rangle\langle j|. \quad (55)$$

As an operator on  $\mathcal{H}_B \otimes \mathbf{C}^{d_A}$ , it has rank at most  $d_A d_B$ , for  $d_B = \dim \mathcal{H}_B$ .

2. Since  $J(\Lambda)$  has rank at most  $d^2$ , it has an eigendecomposition

$$J(\Lambda) = \sum_{i=1}^{d^2} \lambda_i |\psi_i\rangle\langle\psi_i|. \quad (56)$$

Since  $|\psi_i\rangle$  is a bipartite pure state, we know there is an operator  $B_i$  such that

$$|\psi_i\rangle = (I \otimes B_i) |\Omega\rangle. \quad (57)$$

That is,

$$J(\Lambda) = \sum_{i=1}^{d^2} \lambda_i (I \otimes B_i) |\Omega\rangle\langle\Omega| (I \otimes B_i^*). \quad (58)$$

Defining  $A_i = \sqrt{\lambda_i} B_i$ ,

$$J(\Lambda) = \sum_{i=1}^{d^2} (I \otimes A_i) |\Omega\rangle\langle\Omega| (I \otimes A_i^*). \quad (59)$$

Now,

$$\begin{aligned} \text{tr}[C\Lambda(D)] &= \text{dtr}[J(\Lambda)(C \otimes D^T)] = \sum_{i=1}^{d^2} \text{dtr}[(I \otimes A_i) |\Omega\rangle\langle\Omega| (I \otimes A_i^*) C \otimes D^T] \\ &= \sum_{i=1}^{d^2} \text{dtr}[|\Omega\rangle\langle\Omega| C \otimes A_i^* D^T A_i] \\ &= \sum_{i=1}^{d^2} \text{tr}[C \otimes A_i^* D^T A_i |\Omega\rangle]. \end{aligned}$$

Now, we can use Equation (1.4) of notes 7, which is a trick with the maximally entangled state:  $\langle\Omega|A \otimes B|\Omega\rangle = \frac{1}{d} \text{tr}[A^T B]$  for any two matrices  $A$  and  $B$ . Then

$$\begin{aligned} \text{tr}[C\Lambda(D)] &= \sum_{i=1}^{d^2} \text{tr}[C^T A_i^* D^T A_i] \\ &= \sum_{i=1}^{d^2} \text{tr}[(C^T A_i^* D^T A_i)^T] \\ &= \sum_{i=1}^{d^2} \text{tr}[(A_i)^T D (A_i^*)^T C] \\ &= \sum_{i=1}^{d^2} \text{tr}[(A_i)^T D (A_i^*)^T C] \\ &= \text{tr}[C \sum_{i=1}^{d^2} (A_i)^T D (A_i^*)^T]. \end{aligned}$$

Since  $C$  and  $D$  are arbitrary, we've found that  $\{A_i^T\}$  is a set of Kraus operators for  $\Lambda$ .

**Exercise 10** Consider two quantum operations  $\Lambda_1$  and  $\Lambda_2$  acting on a single qubit, having Kraus representations

$$\Lambda_1(\rho) = \sum_{k=1}^2 A_k \rho A_k^\dagger; \quad \Lambda_2(\rho) = \sum_{k=1}^2 V_k \rho V_k^\dagger,$$

with

$$A_1 = \frac{1}{\sqrt{2}}\sigma_0; \quad A_2 = \frac{1}{\sqrt{2}}\sigma_z$$

and

$$V_1 = |0\rangle\langle 0|; \quad V_2 = |1\rangle\langle 1|.$$

How do the actions of  $\Lambda_1$  and  $\Lambda_2$  differ from each other?

**Solution**

A straightforward calculation of the action of  $\Lambda_1$  and  $\Lambda_2$  on

$$\rho = \begin{pmatrix} a & b - ic \\ b + ic & 1 - a \end{pmatrix}$$

(cf. Ex. 2) shows that

$$\Lambda_1(\rho) = \Lambda_2(\rho) = \begin{pmatrix} a & 0 \\ 0 & 1 - a \end{pmatrix}.$$

Hence, the Kraus representation of a CPTP map is not unique.

**Exercise 11** Show that the following three operators form a POVM.

$$\begin{aligned} E_1 &= \frac{\sqrt{2}}{1 + \sqrt{2}} |1\rangle\langle 1| \\ E_2 &= \frac{\sqrt{2}}{2 + 2\sqrt{2}} (|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \\ E_3 &= \mathbf{1} - E_1 - E_2 \end{aligned}$$

Suppose Alice gives Bob a state prepared in one of the two states  $|\psi_1\rangle = |0\rangle$  or  $|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ . Show that if Bob does a measurement characterized by these POVM elements on the state he receives, he never makes an error of misidentification. Discuss the possible outcomes.

**Solution**

Let us first show that  $\{E_1, E_2, E_3\}$  indeed forms a POVM. First,

$$E_1 + E_2 + E_3 = \mathbf{1} \tag{60}$$

by definition of  $E_3$ . Clearly,  $E_1$  and  $E_2$  are positive. For  $E_3$ , we may write a matrix representation. In the computational basis, writing  $c = \frac{\sqrt{2}}{1 + \sqrt{2}}$ ,

$$E_1 = c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_2 = \frac{c}{2} [|0\rangle\langle 0| - |1\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 1|] = \frac{c}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$E_3 = \mathbf{1} - E_1 - E_2 = \begin{pmatrix} 1 - \frac{c}{2} & \frac{c}{2} \\ \frac{c}{2} & 1 - \frac{3}{2}c \end{pmatrix}$$

From this matrix form, we can compute the eigenvalues of  $E_3$ , and find they are

$$\lambda_{\pm} = \frac{2 - 2c \pm \sqrt{2}c}{2}. \tag{61}$$

Substituting our value of  $c$ , we find the two eigenvalues are 0 and  $c$ . Thus,  $E_3$  is positive semi-definite (since it has all *non-negative* eigenvalues).

The probabilities for the different outcomes given that the system is in either the state  $|\psi_1\rangle = |0\rangle$  or  $|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  can be computed by

$$p_i(j) := \mathbb{P}(\text{measurement outcome is } j \mid \text{the state had index } i) = \langle \psi_i | E_j | \psi_i \rangle,$$

which gives:

$$\begin{aligned} p_1(1) &= 0 & p_2(1) &= \frac{\sqrt{2}}{2 + 2\sqrt{2}} \\ p_1(2) &= \frac{\sqrt{2}}{2 + 2\sqrt{2}} & p_2(2) &= 0 \\ p_1(3) &= \frac{2 + \sqrt{2}}{2 + 2\sqrt{2}} & p_2(3) &= \frac{2 + \sqrt{2}}{2 + 2\sqrt{2}} \end{aligned}$$

Therefore, Bob can infer the state of the system with certainty if he measures either outcome 1 or 2, since the system in the state  $|\psi_1\rangle$  ( $|\psi_2\rangle$ )

will never yield outcome 1 (2). Of course,  $p_1(3) = p_2(3)$  and hence, outcome 3 is indecisive and Bob can only guess the state of the system.

The advantage here is that *sometimes* Bob can say what happened with certainty. Since the two states are not orthogonal, no measurement can perfectly distinguish them, but this scheme has the advantage of allowing Bob to sometimes know with certainty what the state was, and sometimes having it be a 50-50 guess, instead of always having to guess (but being right  $> 50\%$  of the time).