Part III, Lent 2019
Quantum Information Theory

Example Sheet 4 Solutions

Exercise 1. Prove that the Holevo capacity is superadditive,

$$\chi^*(\Lambda_1 \otimes \Lambda_2) \geq \chi^*(\Lambda_1) + \chi^*(\Lambda_2),$$

where Λ_1 and Λ_2 denote quantum channels.

Solution.

$$\chi^*(\Lambda_1 \otimes \Lambda_2) := \max_{\{p_x,\rho_x\}} \Big\{ S\Big(\Lambda_1 \otimes \Lambda_2(\sum_x p_x \rho_x)\Big) - \sum_x p_x S\Big(\Lambda_1 \otimes \Lambda_2(\rho_x)\Big)\Big\}$$

where the maximum is over ensembles $\{p_x, \rho_x\}$ with states $\rho_x \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The maxima over all such distributions is larger than the maxima over the restriction to product distributions $\{p_x q_y\}_{x,y}$ and product states $\{\rho_x^A \otimes \sigma_y^B\}_{x,y}$. Thus,

$$\chi^*(\Lambda_1 \otimes \Lambda_2) \geq \max_{\{p_x q_y, \rho_x^A \otimes \sigma_y^B\}} \Big\{ S\Big((\Lambda_1 \otimes \Lambda_2)(\sum_{xy} p_x q_y \rho_x^A \otimes \sigma_y^B)\Big) - \sum_{xy} p_x q_y S\Big(\Lambda_1 \otimes \Lambda_2(\rho_x^A \otimes \sigma_y^B)\Big)\Big\}. \tag{1}$$

Now, using the additivity of the von Neumann entropy under tensor products, we have

$$\begin{aligned} S\Big((\Lambda_1 \otimes \Lambda_2)(\sum_{xy} p_x q_y \rho_x^A \otimes \sigma_y^B)\Big) &= S\Big(\sum_x p_x \Lambda_1(\rho_x^A) \otimes \sum_y q_y \Lambda_2(\sigma_y^B)\Big) \\ &= S\Big(\sum_x p_x \Lambda_1(\rho_x^A)\Big) + S\Big(\sum_y q_y \Lambda_2(\sigma_y^B)\Big), \end{aligned}$$

and

$$\begin{aligned} \sum_{xy} p_x q_y S\Big((\Lambda_1 \otimes \Lambda_2)(\rho_x^A \otimes \sigma_y^B)\Big) &= \sum_{xy} p_x q_y \Big[S\Big(\Lambda_1(\rho_x^A)\Big) + S\Big(\Lambda_2(\sigma_y^B)\Big)\Big] \\ &= \sum_{xy} p_x q_y S\Big(\Lambda_1(\rho_x^A)\Big) + \sum_{xy} p_x q_y S\Big(\Lambda_2(\sigma_y^B)\Big) \\ &= \sum_x p_x S\Big(\Lambda_1(\rho_x^A)\Big) + \sum_y q_y S\Big(\Lambda_2(\sigma_y^B)\Big). \end{aligned}$$

So the right-hand side of (1) is equal to

$$\max_{\{p_x q_y, \rho_x^A \otimes \sigma_y^B\}} \left\{ S\Big(\sum_x p_x \Lambda_1(\rho_x^A)\Big) - \sum_x p_x S\Big(\Lambda_1(\rho_x^A)\Big) + S\Big(\sum_y q_y \Lambda_2(\sigma_y^B)\Big) - \sum_y q_y S\Big(\Lambda_2(\sigma_y^B)\Big) \right\}$$

which is the same as

$$\max_{\{p_x, \rho_x^A\}} \left\{ S\Big(\sum_x p_x \Lambda_1(\rho_x^A)\Big) - \sum_x p_x S\Big(\Lambda_1(\rho_x^A)\Big) \right\} + \max_{\{q_y, \sigma_y^B\}} \left\{ S\Big(\sum_y q_y \Lambda_2(\sigma_y^B)\Big) - \sum_y q_y S\Big(\Lambda_2(\sigma_y^B)\Big) \right\}$$

i.e. $\chi^*(\Lambda_1) + \chi^*(\Lambda_2)$.

Exercise 2. Let $|\psi\rangle_{ABE}$ be a pure state of a tripartite system ABE. Define the *coherent information*[1] from A to B of ψ to be

$$I_c^{A>B}(\psi) = -S(A|B)_\psi.$$

Here $S(A|B)_\psi$ denotes the conditional entropy of the subsystem A with respect to subsystem B, given that the composite system ABE is in the pure state $|\psi\rangle_{ABE}$. Henceforth we shall omit ψ.

Prove the following identities:

1. $\frac{1}{2}I(A:B) + \frac{1}{2}I(A:E) = S(A)$
2. $\frac{1}{2}I(A:B) - \frac{1}{2}I(A:E) = I_c^{A>B}$

Solution.

1. By definition of the quantum mutual information,

$$\begin{aligned}\frac{1}{2}I(A:B) + \frac{1}{2}I(A:E) &= \frac{1}{2}[S(A) + S(B) - S(AB) + S(A) + S(E) - S(AE)] \\ &= S(A) + \frac{1}{2}[S(B) - S(AB) + S(E) - S(AE)].\end{aligned}$$

   Since $|\psi\rangle_{ABE}$ is pure, $S(B) = S(AE)$ and $S(E) = S(AB)$ (since for any bipartition of the systems ABE, the entropies of both reduced density matrices are equal). Thus, the term in brackets vanishes and we obtain $S(A)$ as desired.

[1] In the lecture notes, we defined the coherent information of a quantum channel corresponding to an input state $I_c(\Lambda, \rho)$. Note that $I_c(\Lambda, \rho) = I_c^{R>Q'}(\rho')$ (c.f. Notes 18).

2. Likewise,

$$\begin{aligned}\frac{1}{2}I(A:B) - \frac{1}{2}I(A:E) &= \frac{1}{2}[S(A) + S(B) - S(AB) - S(A) - S(E) + S(AE)] \\ &= \frac{1}{2}[S(B) - S(AB) - S(E) + S(AE)] \\ &= \frac{1}{2}[S(B) - S(AB) - S(AB) + S(B)] \\ &= S(B) - S(AB) = -S(A|B) = I_c^{A>B}.\end{aligned}$$

Exercise 3. Show that if ρ_{AB} is separable then $I_c^{A>B} \leq 0$.

What implication does this have on the conditional entropy $S(A|B)$?

Solution.

Since any separable state can be written as a convex combination of product pure states, any separable state ρ_{AB} may be written as $\rho_{AB} = \sum_i p_i \psi_A^{(i)} \otimes \phi_B^{(i)}$ for some pure states $\psi_A^{(i)}$ and $\phi_B^{(i)}$. Next, because the quantum conditional entropy is concave[2], the coherent information is convex. Thus

$$I_c^{A>B}(\rho) \leq \sum_i p_i I_c^{A>B}(\psi_A^{(i)} \otimes \phi_B^{(i)}) = \sum_i p_i [S(\phi_B^{(i)}) - S(\psi_A^{(i)} \otimes \phi_B^{(i)})].$$

But each entropy is the entropy of a pure state and thus vanishes. Thus, $I_c^{A>B}(\rho) \leq 0$. Thus conditional entropy is therefore non-negative on separable states.

Exercise 4. Use the HSW theorem to find the product state capacity of the depolarizing channel, Λ, defined by

$$\Lambda(\rho) = p\rho + (1-p)\frac{I}{2}.$$

Solution.

Note we are only dealing with qubits (otherwise Λ would not be trace-preserving). First, note that for any unitary U, we have that

$$\Lambda(U\rho U^*) = pU\rho U^* + (1-p)\frac{I}{2}$$

[2]Exercise 12 of Example Sheet 3

and therefore $\Lambda(U\rho U^*) = U\Lambda(\rho)U^*$.

The HSW theorem gives that the product state capacity of Λ is given by

$$\chi^*(\Lambda) = \max_{\{p_x,\rho_x\}} S(\Lambda(\sum_x p_x\rho_x)) - \sum_x p_x S(\Lambda(\rho_x)).$$

We can reduce to pure state ensembles $\{p_x, \psi_x\}$. Then defining the average state $\rho := \sum_x p_x\psi_x$, we consider

$$S(\Lambda(\rho)) - \sum_x p_x S(\Lambda(\psi_x))$$

Since all pure states can be related by unitaries, we have that for each x and y, there is a unitary U such that $\psi_x = U\psi_y U^*$. Thus,

$$S(\Lambda(\psi_x)) = S(\Lambda(U\psi_y U^*)) = S(U\Lambda(\psi_y)U^*) = S(\Lambda(\psi_y))$$

using that the entropy is unitarily invariant. Thus, $S(\Lambda(\psi_x))$ does not depend on x. To calculate its value, we can extend $|\psi_x\rangle$ to an orthonormal basis given by $\{|\psi_x\rangle, |\psi_x\rangle^\perp\}$. Then in this basis,

$$\Lambda(\psi_x) = p\psi_x + (1-p)\frac{I}{2} = \begin{pmatrix} p + \frac{1-p}{2} & 0 \\ 0 & \frac{1-p}{2} \end{pmatrix}.$$

Thus, $S(\Lambda(\psi_x)) = h\left(\frac{1-p}{2}\right)$ where h is the binary entropy. Since $\sum_x p_x = 1$, we have

$$S(\Lambda(\rho)) - \sum_x p_x S(\Lambda(\psi_x)) = S(\Lambda(\rho)) - h\left(\frac{1-p}{2}\right).$$

It remains to maximize the output entropy $S(\Lambda(\rho))$ over all states ρ. Since Λ is unitarily invariant, we can work in the eigenbasis of ρ, in which case $\rho = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$ for some number $0 \le \lambda \le 1$. In this basis, we have

$$\Lambda(\rho) = \begin{pmatrix} p\lambda + \frac{1-p}{2} & 0 \\ 0 & p(1-\lambda) + \frac{1-p}{2} \end{pmatrix}.$$

Thus,

$$S(\Lambda(\rho)) = h\left(p\lambda + \frac{1-p}{2}\right) = h\left(\frac{1}{2} + p(\lambda - \frac{1}{2})\right)$$

Since the binary entropy is maximized at $\frac{1}{2}$ at which point it takes the value $\log_2 2 = 1$ (since, e.g. the entropy is maximized on a uniform distribution), $S(\Lambda(\rho))$ is maximized when $\lambda = \frac{1}{2}$. Thus,

$$\chi^*(\Lambda) = 1 - h\left(\frac{1-p}{2}\right)$$

gives the product state capacity of the depolarizing (qubit) channel.

Exercise 5. Alice prepares a photon in one of two polarization states, given by the kets $|a\rangle := |1\rangle$, and $|b\rangle := \sin\theta|0\rangle + \cos\theta|1\rangle$, depending on the outcome of a fair coin toss. If the outcome is heads, she prepares the state $|a\rangle$. Otherwise she prepares the state $|b\rangle$. Evaluate the Holevo χ quantity for her ensemble of states. (Use the convention that $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ \ 1)^T$). For what value of θ is the Holevo bound achieved? Explain why.

Solution.

The average state is given by

$$\begin{aligned}\rho &= \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|b\rangle\langle b| \\ &= \frac{1}{2}[|1\rangle\langle 1| + \sin^2\theta|0\rangle\langle 0| + \cos^2\theta|1\rangle\langle 1| + \sin\theta\cos\theta[|0\rangle\langle 1| + |1\rangle\langle 0|]] \\ &= \frac{1}{2}\begin{pmatrix}\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & 1+\cos^2\theta\end{pmatrix}\end{aligned}$$

which has eigenvalues $\frac{1}{2}(1 \pm \cos\theta)$. Since the two states are pure, each has zero entropy. The Holevo quantity is therefore simply $h(\frac{1}{2}(1 \pm \cos\theta))$ where h is the binary entropy. Since the binary entropy is maximized at $\frac{1}{2}$, this quantity is maximized when $\cos\theta = 0$, i.e. $\theta = \frac{\pi}{2} + \pi z$ for any $z \in \mathbb{Z}$.

Exercise 6. Alice encodes classical information into n photons which she sends to Bob through a quantum channel. What is the maximum number of bits of information that Bob can infer from the output of the channel by doing measurements on it?

Hint: Use the Holevo bound.

Solution.

Let X denote Alice's classical information, and Y the random variable obtained by Bob from measurements on what he obtains from the quantum channel. The amount of bits of information Bob can infer about Alice's information is exactly $I(X:Y)$. The Holevo bound gives that

$$I(X:Y) \leq \chi(\{p_x, \rho_x\})$$

where $\{p_x, \rho_x\}$ is the ensemble used by Alice to encode her information. We don't know how the encoding is done or if the channel is noisy, but we know only n photons were used, which we assume are qubits (information encoded only in the polarization). Thus, the output state $\rho = \sum_x p_x \rho_x$ is 2^n-dimensional, and its entropy is at most n. Since $\chi(\{p_x, \rho_x\}) \leq S(\rho) \leq n$, we obtain an upper bound of n.

Exercise 7. Entropy exchange. The entropy exchange for a state ρ and a quantum channel Λ is defined as follows:

$$S(\rho, \Lambda) := S(\rho'_{RQ}),$$

where $\rho'_{RQ} = (\mathrm{id}_R \otimes \Lambda)\psi^\rho_{RQ}$, with ψ^ρ_{RQ} being a purification of ρ.

1. Prove that $S(\rho, \Lambda) = S(\rho'_E)$, where $\rho'_E = \mathrm{tr}_{RQ}(\rho'_{RQE})$ with

   $$\rho'_{RQE} = (I_R \otimes U_{QE})(\psi^\rho_{RQ} \otimes |0_E\rangle\langle 0_E|)(I_R \otimes U^\dagger_{QE}),$$

   with U_{QE} being the Stinespring dilation of the channel Λ.

   Thus the entropy exchange can be interpreted as the amount of entropy introduced by Λ into an initially pure environment.

2. Prove that the entropy exchange can also be written in the form

   $$S(\rho, \Lambda) = S(W) = -\operatorname{tr} W \log W,$$

   where W denotes a matrix with elements $W_{ij} = \mathrm{tr}(A_i \rho A_j^\dagger)$, where $\{A_i\}$ denote a set of Kraus operators of Λ.

Solution.

1. We can see that indeed, $\rho'_{RQ} = \mathrm{tr}_E\, \rho'_{RQE}$. Moreover, ρ'_{RQE} is a pure state as $\psi^\rho_{RQ} \otimes |0_E\rangle\langle 0_E|$ is pure, and we conjugate by a unitary, $I_R \otimes U_{QE}$. Thus, the Schmidt decomposition tells us that across any bipartite partition, the reduced density matrices have the same non-zero eigenvalues, and thus the same entropy. That is, $S(\rho'_{RQ}) = S(\rho'_E)$ as desired.

2. We can trace out R to find

$$\rho'_E = \mathrm{tr}_Q[U_{QE}(\rho_Q \otimes |0_E\rangle\langle 0_E|)U_{QE}^\dagger].$$

Now, we can choose our Strinespring representation to act as $U|\psi_A\rangle \otimes |0_E\rangle = \sum_i A_i|\psi_A\rangle \otimes |i\rangle$ for any pure state $|\psi_A\rangle$ (as shown in the notes where we construct U), where $\{A_i\}$ is a Kraus representation of Λ. Now, let $\rho_Q = \sum_\alpha \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ be a decomposition into pure states. Then

$$\begin{aligned}\rho'_E &= \sum_\alpha \lambda_\alpha \,\mathrm{tr}_Q[U_{QE}(|\psi_\alpha\rangle\langle\psi_\alpha| \otimes |0_E\rangle\langle 0_E|)U_{QE}^\dagger] \\ &= \sum_\alpha \lambda_\alpha \sum_{i,j} \mathrm{tr}_Q[A_i|\psi_\alpha\rangle\langle\psi_\alpha|A_j^\dagger \otimes |i_E\rangle\langle j_E|] \\ &= \sum_\alpha \lambda_\alpha \sum_{i,j} \mathrm{tr}[A_i|\psi_\alpha\rangle\langle\psi_\alpha|A_j^\dagger]|i_E\rangle\langle j_E| \\ &= \sum_{i,j} \mathrm{tr}[A_i\rho_Q A_j^\dagger]|i_E\rangle\langle j_E|.\end{aligned}$$

Therefore, $W := \rho'_E$ has a matrix representation where the matrix elements are given by $\mathrm{tr}[A_i\rho_Q A_j^\dagger]$. By Step (1), this completes the proof.

Exercise 8. Quantum Fano inequality. Prove the quantum Fano inequality:

$$S(\rho,\Lambda) \leq h(F_e(\rho,\Lambda)) + (1 - F_e(\rho,\Lambda))\,\log(d^2 - 1)$$

where

- $S(\rho,\Lambda)$ is the entropy exchange defined in the previous question,
- Λ denotes a quantum operation with Kraus representation

$$\Lambda(\rho) = \sum_{i=0}^{d^2} V_i \rho V_i^\dagger,$$

  with ρ being the state of a quantum system Q with Hilbert space of dimension d.
- $h(\cdot)$ denotes the binary Shannon entropy, i.e., for any $0 < p < 1$,

$$h(p) = -p\log p - (1-p)\log(1-p).$$

- $F_e(\rho, \Lambda)$ defined by

$$F_e(\rho, \Lambda) := \langle \Psi^\rho_{RQ} | (\mathrm{id} \otimes \Lambda) \Psi^\rho_{RQ} | \Psi^\rho_{RQ} \rangle,$$

  is the entanglement fidelity. Here $|\Psi^\rho_{RQ}\rangle$ is a purification of the state ρ, with R denoting the reference system used for the purification.

What is the implication of the quantum Fano inequality as regards entanglement?

Solution.

First, we have the useful result that the entropy is increasing under unital maps. That is, if Λ is a unital CPTP map, then $S(\Lambda(\rho)) \geq S(\rho)$ for any state ρ. We can see this simply by using the data-processing inequality:

$$S(\rho) = -D(\rho \| I) \leq -D(\Lambda(\rho) \| \Lambda(I)) = -D(\Lambda(\rho) \| I) = S(\Lambda(\rho)).$$

Now, from the previous exercise, we know that $S(\rho, \Lambda) = S(\rho'_{RQ})$ for $\rho'_{RQ} = (\mathrm{id}_R \otimes \Lambda)(\psi^\rho_{RQ})$. Let us consider an orthonormal basis $\{|f_i\rangle\}_{i=1}^{d^2}$ for the joint space $\mathcal{H}_R \otimes \mathcal{H}_Q$ such that $|f_1\rangle = |\psi^\rho_{RQ}\rangle$. That is, we take the first vector to be $|\psi^\rho_{RQ}\rangle$ and then use Grahm-Schmidt to complete to an orthonormal basis. Then we define the unital CPTP map

$$\Gamma(\omega_{RQ}) = \sum_i \langle f_i | \omega_{RQ} | f_i \rangle \, |f_i\rangle\langle f_i|.$$

This is sometimes called a *pinching map* for the basis $\{|f_i\rangle\}$. We can check it is CPTP as it has Kraus operators $\{|f_i\rangle\langle f_i|\}_{i=1}^{d^2}$, and unital by substituting $\omega_{RQ} = I_{RQ}$. By our "useful result", we have

$$S(\rho, \Lambda) = S(\rho'_{RQ}) \leq S(\Gamma(\rho'_{RQ})) = H(\{\langle f_i | \rho'_{RQ} | f_i \rangle\}_{i=1}^{d^2})$$

which is the Shannon entropy of the probability distribution $p := \{\langle f_i | \rho'_{RQ} | f_i \rangle\}_{i=1}^{d^2}$ (as these are the eigenvalues of $\Gamma(\rho'_{RQ})$ which can see by the definition of Γ). Moreover, we know the first entry is $\langle f_i | \rho'_{RQ} | f_i \rangle = F_e(\rho, \Lambda)$ by construction. Thus, if we define $\varepsilon := 1 - F_e(\rho, \Lambda)$, the distribution p has probability $1 - \varepsilon$ for the first entry. Therefore, by classical Fano's inequality[3],

$$H(p) \leq h(\varepsilon) + \varepsilon \log(d^2 - 1) = h(1 - \varepsilon) + \varepsilon \log(d^2 - 1)$$

[3]Exercise 8 of Example sheet 1

where to obtain the equality we use that the binary entropy is symmetric (in the sense that $h(x) = h(1-x)$ for $x \in [0,1]$). Substituting ε yields thus result.

By the previous exercise, we can see $S(\rho, \Lambda)$ as the entropy of $\mathrm{tr}_E[\rho'_{RQE}]$ (and also of $\mathrm{tr}_{RQ}[\rho'_{RQE}]$). Since ρ'_{RQE} is a pure state, this quantity is the *entropy of entanglement* of the state ρ'_{RQE} across the partition $RQ|E$. This is an *entanglement monotone* (i.e. non-increasing under local operations and classical communication, so-called LOCC operations), and you can check that it is zero for product states and non-zero for entangled states. Since we can see ρ'_{RQE} as induced by the action of Λ (in Stinespring form) on a product state $\psi^\rho_{RQ} \otimes |0_E\rangle\langle 0_E|$, the quantity $S(\rho, \Lambda)$ is a measurement of the entanglement generated by Λ by its action on this state. On the other hand, $F_e(\rho, \Lambda)$ is the fidelity between ψ^ρ_{RQ} and $\Lambda(\psi^\rho_{RQ})$, which is close to 1 when ψ^ρ_{RQ} is close to $\Lambda(\psi^\rho_{RQ})$ (so ε is close to zero in this case). So we can upper bound the entanglement generated by Λ in terms of how much it changes the state in fidelity (and in particular, if Λ does not change the state at all, then it can't generate any entanglement).

Exercise 9. Let ρ be a quantum state on $\mathcal{H}_Q$, and Λ a quantum operation on Q. We say it is possible to *perfectly reverse* Λ on ρ if there exists a quantum operation Λ' on Q such that $F_e(\Lambda' \circ \Lambda, \rho) = 1$. Show that if it is possible to perfectly reverse Λ on ρ, then

$$S(\rho) = I_c(\Lambda, \rho)$$

where $I_c(\Lambda, \rho)$ is the coherent information of the channel Λ when the input state is ρ.

Note: there was an error in the stated version of this question; $\Lambda' \circ \Lambda(\rho) = \rho$ *is not enough to prove that* $S(\rho) = I_c(\Lambda, \rho)$.

Solution.
The quantum data processing inequality gives that

$$I_c(\Lambda' \circ \Lambda, \rho) \leq I_c(\Lambda, \rho) \leq S(\rho_Q). \tag{2}$$

Let Λ have a Stinespring unitary U_{QE_1} and Λ' have a Stinespring unitary U'_{QE_2}. Then if ψ_{RQ} is a purification of ρ_Q, let

$$\rho''_{RQE_1E_2} = U'_{QE_2} U_{QE_1} (\psi_{RQ} \otimes |0\rangle\langle 0|_{E_1} \otimes |0\rangle\langle 0|_{E_2}) U^\dagger_{QE_1} U'^\dagger_{QE_2}$$

denote the joint state of RQE_1E_2 after the action of both channels. Note that identity operators have been omitted in the above expression; e.g. U_{QE_1} means $U_{QE_1} \otimes I_{RE_2}$. Note that $\rho''_{RQE_1E_2}$ is is a pure state. From the definition of coherent information, we have

$$I_c(\Lambda' \circ \Lambda, \rho) = S(\rho''_Q) - S(\rho''_{E_1E_2}). \tag{3}$$

From Exercise 7, we know that

$$S(\rho''_{E'E}) = S(\rho, \Lambda' \circ \Lambda)$$

is the entropy exchange, and by the monotoncity of the fidelity under partial trace, we can see that

$$1 = F_e(\Lambda' \circ \Lambda, \rho) \leq F(\rho''_Q, \rho_Q) \leq 1$$

and hence $\rho''_Q = \rho_Q$. Thus, the right-hand side of (3) becomes

$$S(\rho_Q) - S(\rho, \Lambda' \circ \Lambda).$$

Since

$$0 \leq S(\rho, \Lambda' \circ \Lambda) \leq h(F_e(\rho, \Lambda)) + (1 - F_e(\rho, \Lambda)) \log(d^2 - 1) = 0$$

by the quantum Fano inequality, we have $S(\rho, \Lambda' \circ \Lambda) = 0$ and hence $I_c(\Lambda' \circ \Lambda, \rho) = S(\rho_Q)$. Thus the DPI (2) yields the result.